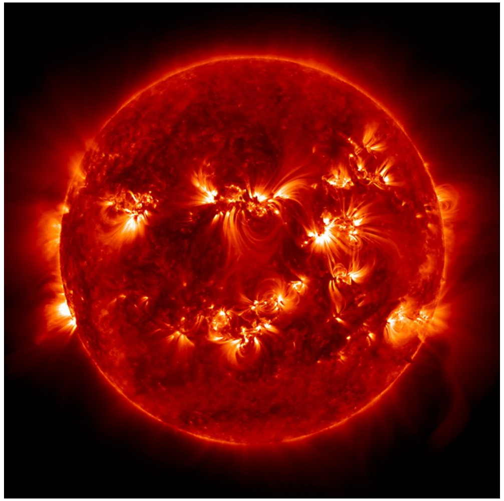
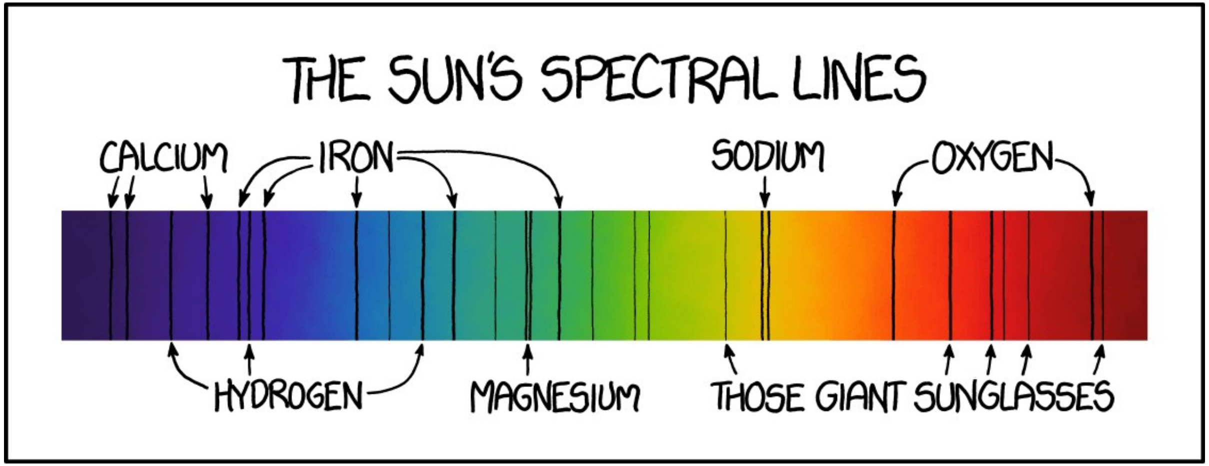
PHYS 7810 / COLLAGE2021:
Solar Spectral Line Diagnostics





Lecture 03: Solving Radiative Transfer Equation

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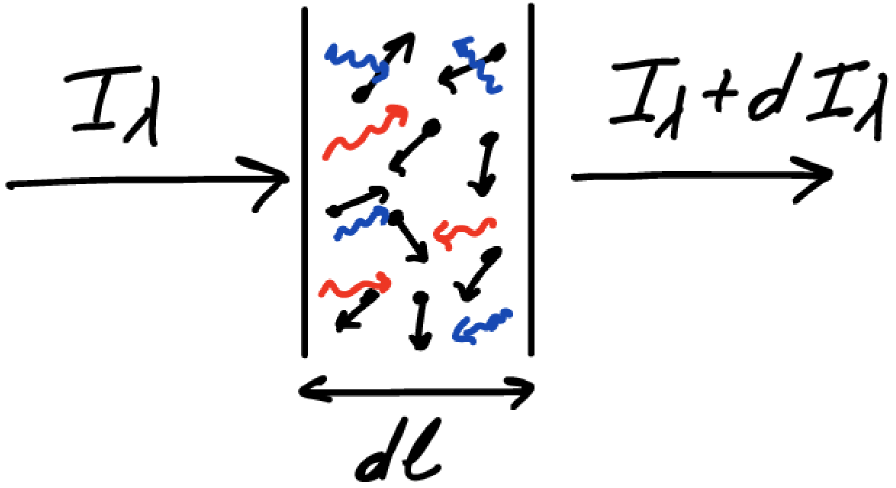
Plan for today:

- We pick up where we left off last time!
- **Solve RTE** for some simplified cases and study behavior of the solution
- **Make a connection to the atmosphere of the Sun:** where and how do we solve RTE, with which boundary conditions, etc.
- **Hint the importance of wavelenght dependence**

Radiative Transfer Equation

- A differential equation that describes how the intensity changes over a given path (ray, pencil, direction):

Inside of this infinitesimal "slab", we have absorption and emission processes



This is the simplest way to express radiative transfer equation in the "along-the-ray" form: It basically has 1D spatial dependence and, since we fixed the direction, no angular dependence. Wavelength dependence is, of course, still there.

$$\frac{dI_\lambda}{dl} = j_\lambda - \chi_\lambda I_\lambda$$

Understanding coefficient of emission

If we focus on the “emitted” part of the intensity, we would get:

$$\frac{dI_\lambda^{\text{emitted}}}{dl} = j_\lambda$$

So, the emissivity quantifies amount of intensity added per unit length.

- Now, we can unpack it further:

$$j_\lambda = \frac{dE^{\text{emitted}}}{dS\, dl\, dt\, d\Omega\, d\lambda} = \frac{dn^{\text{emitted}} hc/\lambda}{dt\, d\Omega\, d\lambda}$$

Which is why sometimes you will find it called “volume emission coefficient”

The absorption coefficient (opacity)

- We can have a very similar story here:

$$\frac{dI_\lambda^{\mathrm{abs}}}{dl} = -\chi_\lambda I_\lambda$$

- Note the different form: the absorbed intensity is proportional to the (input) intensity. Does that make sense to you? Any analogies? (2 mins)
- The intensity should have units of inverse length. What if we relate it somehow to number density of absorption events (absorbers?)

$$\chi_\lambda[m^{-1}] = n^{\mathrm{absorbers}}[\mathrm{m}^{-3}] \times \sigma[\mathrm{m}^2]$$

- This is more “classical” than the previous argument, but it very intuitively tells us what opacity depends on!

Mean free path

- More opacity, shorter path photons can traverse before being absorbed (or, generally, removed from the ray)
- This means, more opaque the medium is more “trapped” the photons are
- This helps establish “local” thermodynamic equilibrium: photons are trapped in a thin layer and they establish equilibrium with the matter
- Keep in mind that the mean free path is also wavelength dependent

$$\bar{l}_\lambda = \frac{1}{\chi_\lambda}$$

Solving Radiative Transfer Equation

- Assuming that our analytical / numerical powers are infinite, what do we need to solve RTE?
- You extend your line-of-sight to the edge of the object.
- You specify the boundary condition (unless you can assume the object is semi-infinite)
- You solve the differential equation, and go publish some nice papers.

Solving radiative transfer equation

- This is very reasonable and understandable form:

$$\frac{dI_\lambda}{dl} = j_\lambda - \chi_\lambda I_\lambda$$

- Let’s make it worse and less understandable and divide both sides with opacity (or if you want, minus opacity):

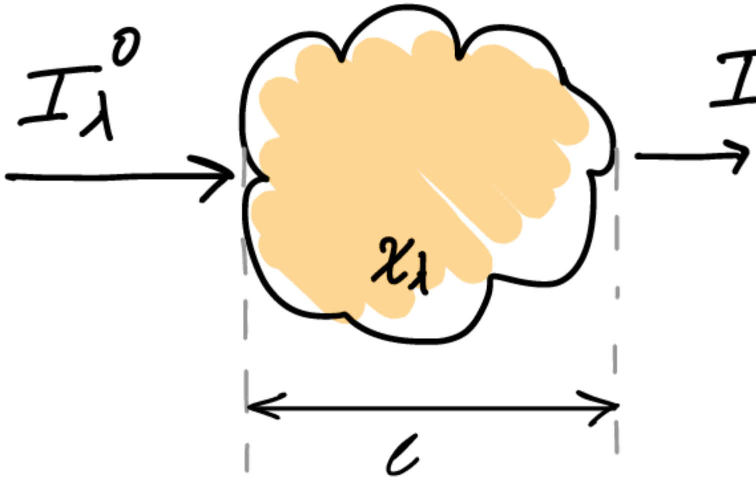
$$\frac{dI_\lambda}{\chi_\lambda dl} = I_\lambda - \frac{j_\lambda}{\chi_\lambda}$$

- And let’s see what we get here

Optical depth

- What we did is we changed variable from geometrical path, that is insensitive to optical properties of the medium to a dimensionless quantity that depends on the opacity of the medium (and the wavelength).

- Example, a homogeneous cloud that only absorbs:



$I_\lambda^+ = ?$

$$\frac{dI_\lambda}{dl} = -\chi_\lambda I_\lambda$$

$$\frac{dI_\lambda}{I_\lambda} = -\chi_\lambda \cdot dl$$

$$\int_{I_\lambda^0}^{I_\lambda^+} \frac{1}{I_\lambda} dI_\lambda = \int_0^l -\chi_\lambda dl$$

$$\ln I_\lambda^+ - \ln I_\lambda^0 = \underbrace{-\chi_\lambda l}_{\tau_\lambda} \rightarrow I_\lambda^+ = I_\lambda^0 \cdot e^{-\tau_\lambda}$$

Note that, by looking at original and emergent intensity, we cannot tall whether it's a small dense cloud or a large tenuous cloud

Source function

- We also define another new quantity, **the Source function**, as the ratio between emission and absorption coefficients.

$$S_\lambda = \frac{j_\lambda}{\chi_\lambda}$$

- Radiative Transfer equation becomes:

$$\frac{dI_\lambda}{d\tau_\lambda} = S_\lambda - I_\lambda$$

- We have a new form. Independent coordinate depends on the wavelength. But the solution will be very simple

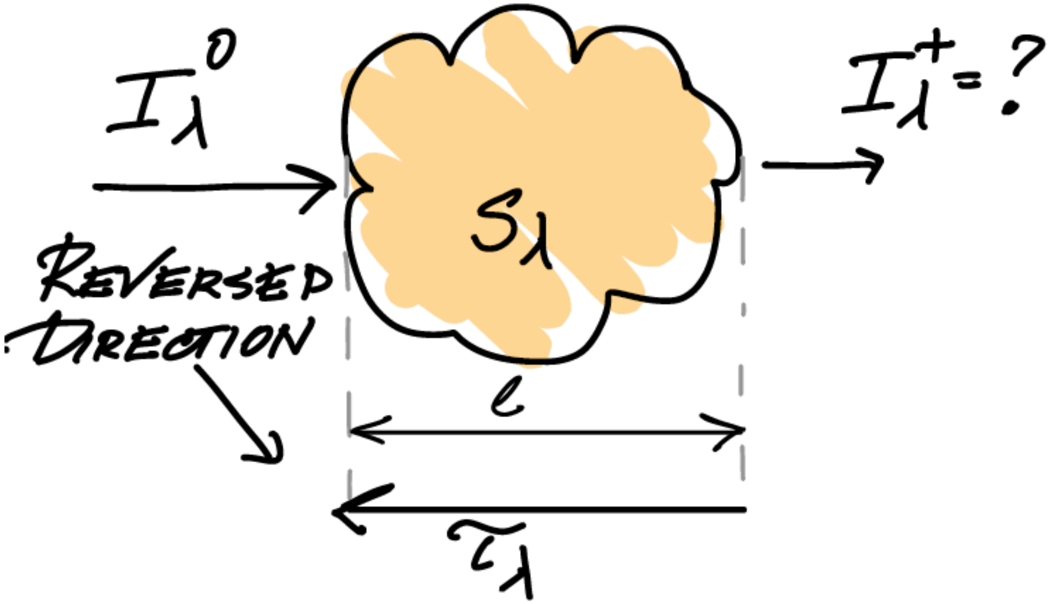
Some notes on the source function

$$\frac{dI_\lambda}{d\tau_\lambda} = S_\lambda - I_\lambda \qquad S_\lambda = \frac{j_\lambda}{\chi_\lambda}$$

- Source function has the same units as the intensity.
- One way to think about it is: **intensity added to the ray per unit optical path**
- People often say that optical depth is a natural coordinate for radiative transfer. It is because from the outside we cannot tell the geometrical scale (think about earlier argument about the cloud)

Formal solution

- We will often want to reverse the direction of optical depth so that 0 is at the top. This leads to the form you are maybe most familiar with:



$$\frac{dI_\lambda}{d\tau_\lambda} = I_\lambda - S_\lambda$$

With a solution:

$$I_\lambda^+ = I_\lambda^0 e^{-\tau_\lambda} + \int_0^{\tau_\lambda} S(t)e^{-t}dt$$

Formal solution

- What do we need to be able to solve this integral?

$$I_\lambda^+ = I_\lambda^0 e^{-\tau_\lambda} + \int_0^{\tau_\lambda} S(t) e^{-t} dt$$

- Incident intensity (boundary condition), we need run of S on the optical depth and for that we need the opacity.
- It is called formal solution because it assumes we can explicitly calculate opacity and emissivity (emission coefficient). This is not always true.
- What are some things you can conclude from the formal solution? (3 mins)

Some things that came to my mind

$$I_\lambda^+ = I_\lambda^0 e^{-\tau_\lambda} + \int_0^{\tau_\lambda} S(t) e^{-t} dt$$

- More optically thick (deep) the object is, less the incident intensity matters.
- The source function closer to the surface (t=0), matters more.
- Actually, Source function in great depths matters very little.
- It’s all exponentials, so things really fall of quick for monochromatic optical depth above “a few”
- Everything is wavelength dependent and wavelengths are de-coupled, which means we can solve this wavelength-by-wavelength.

Simplified solutions – constant source function

- Let's assume the cloud is indeed homogeneous and the source function is constant everywhere

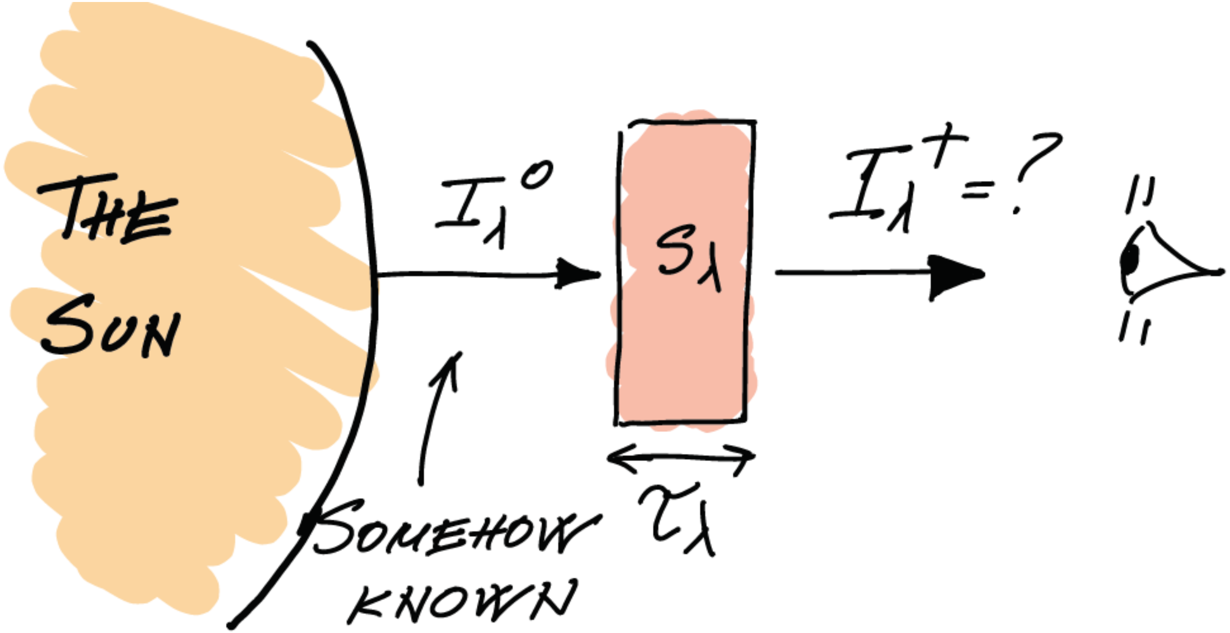
$$I_\lambda^+ = I_\lambda^0 e^{-\tau_\lambda} + \int_0^{\tau_\lambda} S(t) e^{-t} dt$$

$$I_\lambda^+ = I_\lambda^0 e^{-\tau_\lambda} + S_\lambda (1 - e^{-\tau_\lambda})$$

- Emergent intensity is weighted contribution of the incoming intensity and the source function. The weighting depends on the optical depth of the object.

Practical application:

- We can use this equation to model radiative transfer through any a of slab (say, a filament?) that can be assumed to be constant
(either because of physics or simply because we do not know better)



$$I_\lambda^+ = I_\lambda^0 e^{-\tau_\lambda} + S_\lambda (1 - e^{-\tau_\lambda})$$

Simplified solutions – linear source function

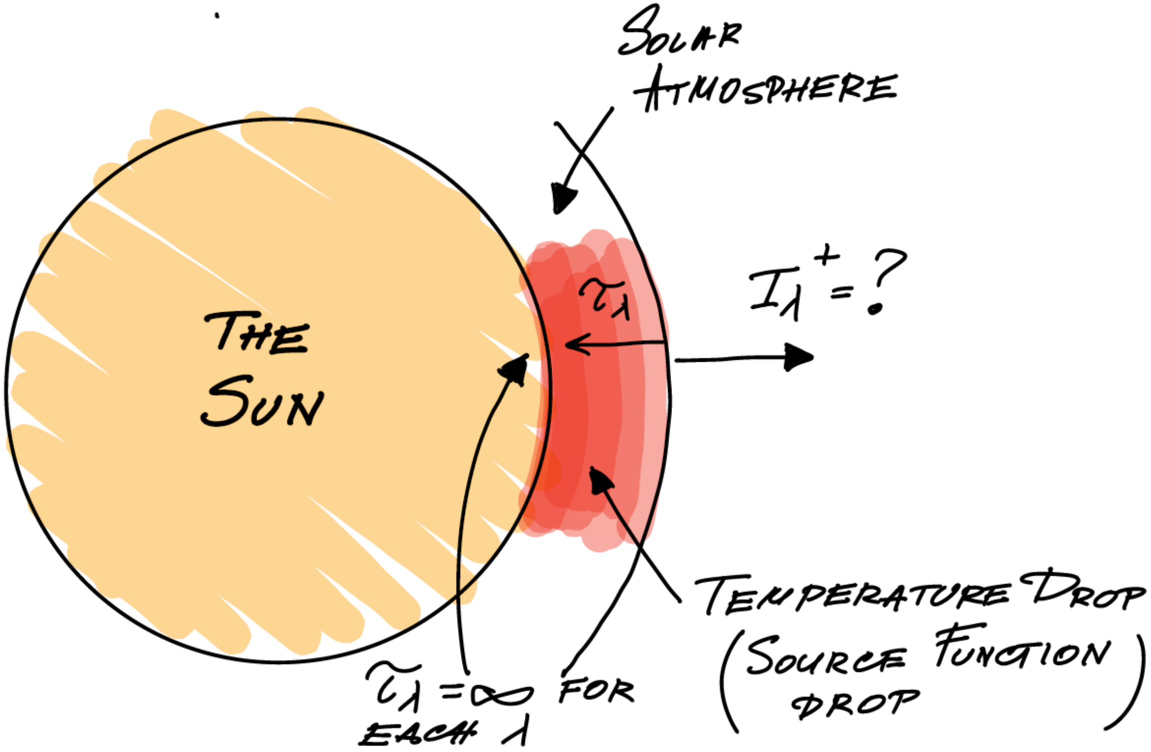
- Let's assume that source function linearly increases with optical depth
- In addition, it is common to assume that the optical depth is very large (infinite). We will use this to model solar atmosphere later

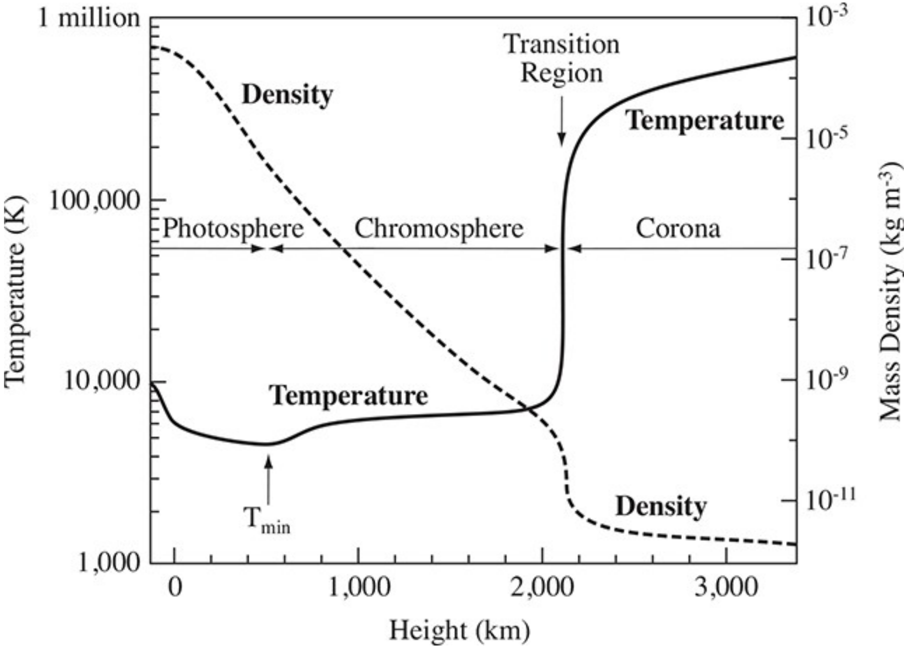
$$I_\lambda^+ = I_\lambda^0 e^{-\tau_\lambda} + \int_0^{\tau_\lambda} (a + bt) e^{-t} dt$$

$$I_\lambda^+ = a + b = S(\tau_\lambda = 1)$$

- The emergent intensity is approximately equal to Source function at the optical depth unity *at corresponding wavelength:* ***Eddington – Barbier relationship.***

Practical application – modeling solar photosphere

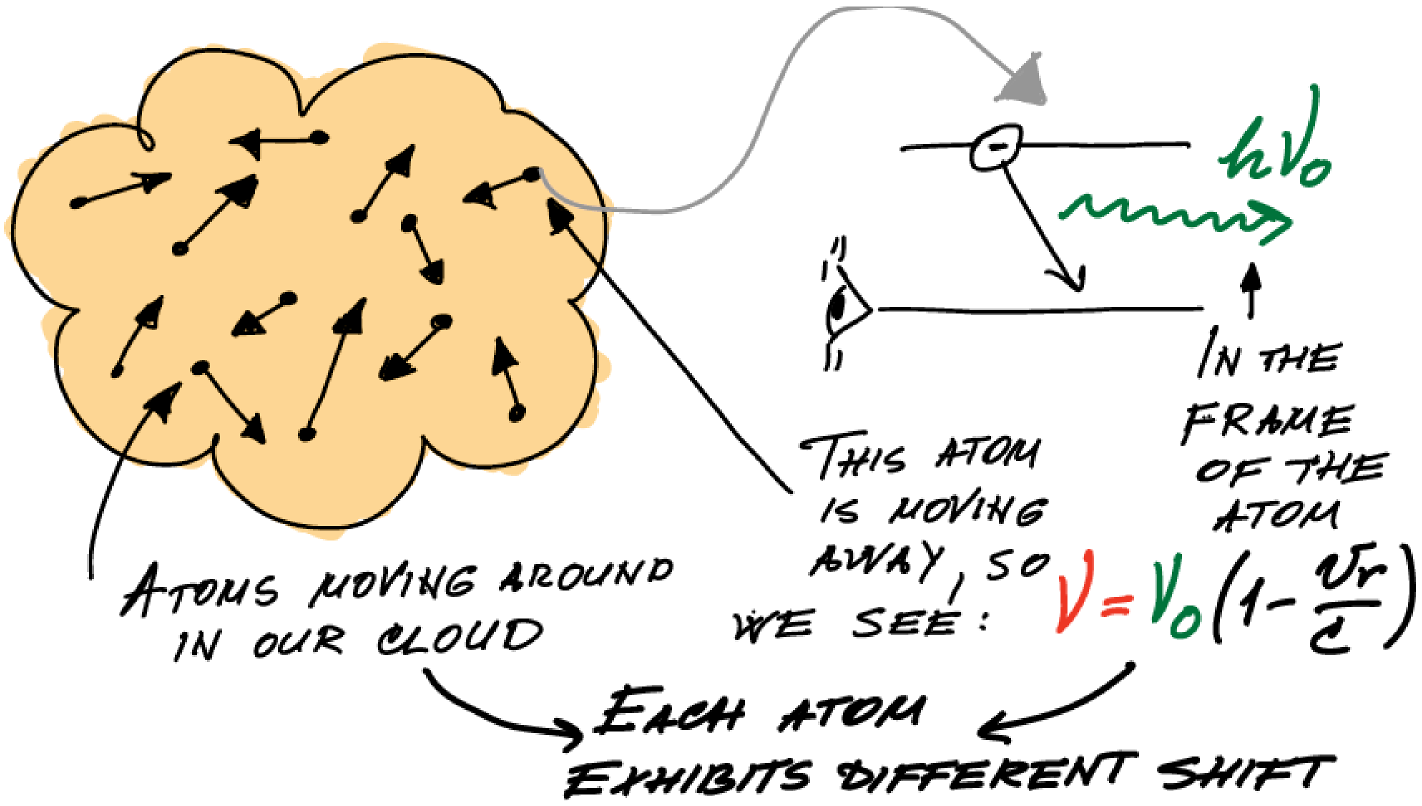




Wavelength scaling

- Believe it or not, the optical depth is more dramatically scaling with wavelength than the source function
- The reason for that is that optical depth depends on the opacity (absorption coefficient) only, while the source function depends on the ratio, so wavelength dependence somewhat "cancels"
  (next class we will see that this is actually super true)
- Let's very quickly make some assumptions about the wavelength dependence of the opacity and emissivity due to spectral line processes

Simple spectral line broadening



The line-of-sight velocity distribution follows Gaussian distribution. Then, when we sum up emissions (absorptions) by all the particles, Doppler-shifted frequency (and wavelength) will follow Gaussian Distribution too!

Dependence of opacity on the wavelength

- We are going to assume that there is some amount of “flat” opacity. Think about this as a fog, that equally absorbs at all the wavelengths.
- This is the so called “continuum” opacity. That is, opacity arising from all the non-spectral line sources.
- On top of that there is sharply varying spectral line opacity. We model it as a constant times a Gaussian function, so:

$$\chi_\lambda = \chi_{\mathrm{c}} + \chi_{\mathrm{l}} \times \phi(\lambda)$$

$$\phi_\lambda = \frac{1}{\sqrt{\pi}\Delta_\lambda} e^{-(\lambda-\lambda_0)^2/\Delta_\lambda^2}$$

Dependence of the optical depth and the source function on the wavelength

- Now we assume that the “profile” is constant in space, we should get:

$$\tau_\lambda = \tau_c(1 + k\,\phi(\lambda))\ ;\ k = \chi_{\rm l}/\chi_{\rm c}$$

What about the source function?

- It’s reasonable to assume that velocity distribution is the same regardless in which state the atom is in, and that the profile is the same:

$$S_\lambda = \frac{j_\lambda}{\chi_\lambda} = \frac{j_{\rm c} + j_{\rm l}\phi_\lambda}{\chi_{\rm c} + \chi_{\rm l}\phi_\lambda} = S$$

It actually requires these two ratios to be the same, but trust me (and Kirchoff), that in certain regimes they are

Ok, jupyter notebook time!

TO-DO: Plot Solutions of the RTE for these simple cases for different optical depths and different wavelengths