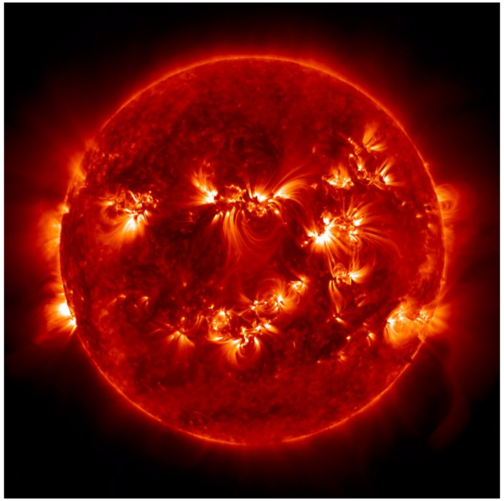
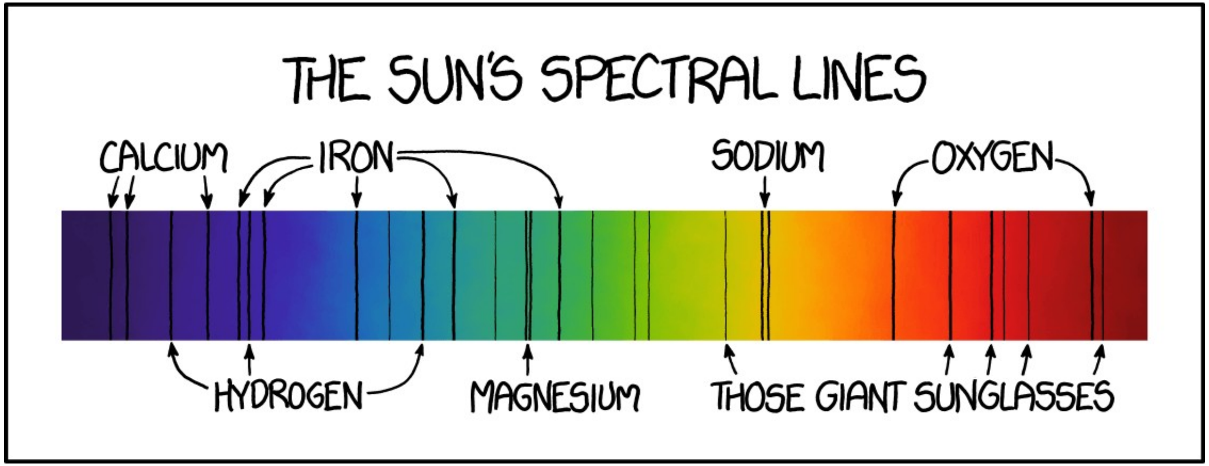
PHYS 7810 / COLLAGE2021:
Solar Spectral Line Diagnostics





Lecture 02: Basic Radiative Transfer

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Where we left off last time...

Coefficient of the emission of the medium

$$\frac{dI(\vec{r}, \mathbf{n}, \lambda)}{dl} = j(\vec{r}, \mathbf{n}, \lambda) - \chi(\vec{r}, \mathbf{n}, \lambda) I(\vec{r}, \mathbf{n}, \lambda)$$

Change of our quantity of interest (intensity), over the elementary geometrical path.

Coefficient of the absorption of the medium

Plan for today:

- **Define formally and understand intuitively** all the quantities appearing in the radiative transfer equation (RTE)
- **Solve RTE** for some simplified cases and study behavior of the solution
- **Make a connection to the atmosphere of the Sun:** where and how do we solve RTE, with which boundary conditions, etc.
- **Hint the importance of wavelenght dependence**

Again, RT is combination of small and big scales

This whole equation tells us how the things behave on big scales. (ds is geometrical length, and light propagates over very large distances inside of astrophysical objects.

$$\frac{dI(\vec{r}, \mathbf{n}, \lambda)}{dl} = j(\vec{r}, \mathbf{n}, \lambda) - \chi(\vec{r}, \mathbf{n}, \lambda) I(\vec{r}, \mathbf{n}, \lambda)$$

These two coefficients, on the other side, depend on specific physics of emission and absorption

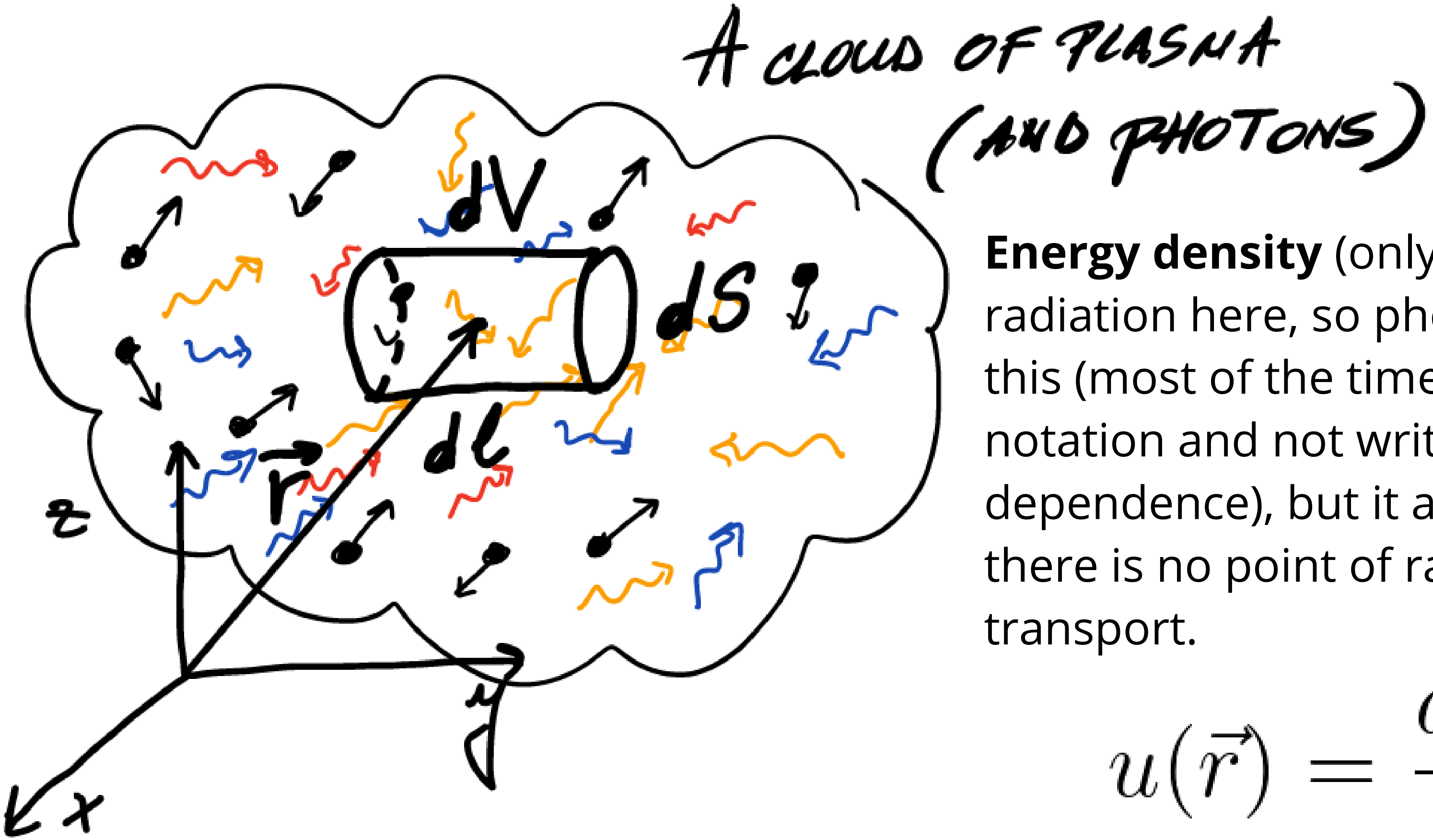
For the moment, we are **defining** the quantities

- Which means, we won't concern ourselves with *why* absorption and emission happen, we will concern ourselves with *what they are* (in context of quantities)
- But, just to flex our brains a bit, let's list some emission and absorption processes (try to think as microscopic as possible)… (2 mins + 2 mins discussion)

Specific Monochromatic Intensity

- This is our quantity of interest. We want to calculate (specific monochromatic) intensity at the top of the solar atmosphere (or emerging from any other object in general), and to compare it with what we measure.
- Said in words it describes **angular, temporal and spectral distribution of energy transported through a specific surface.**
- This means that intensity depends on:
  location (3 coordinates)
  time (1 coordinate, this one we will for now completely neglect)
  angle / direction (2 coordinates)
  wavelength (1 coordinate)
- Now let's convince ourselves of this!

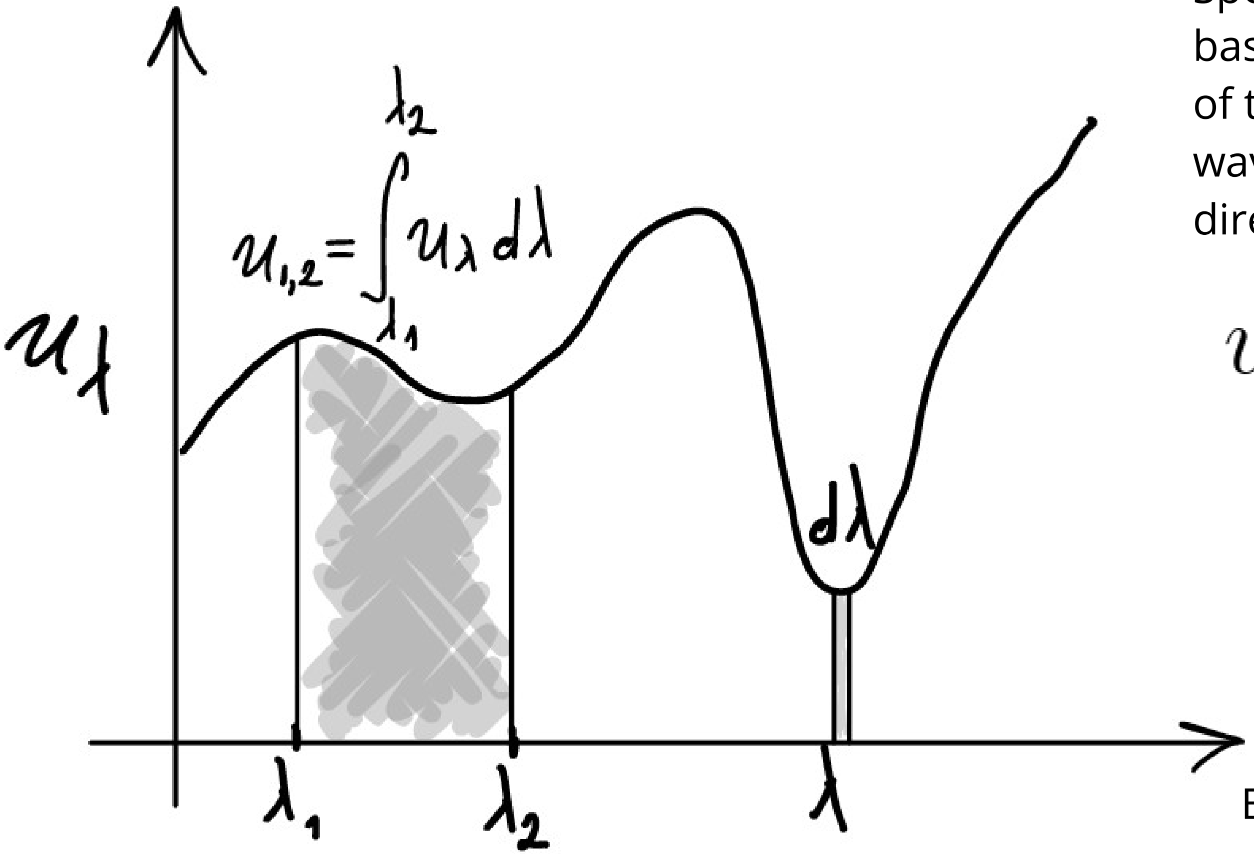
Step-by-step toward the definition of intensity



Energy density (only talking about radiation here, so photons), looks like this (most of the time I will shorten the notation and not write spatial dependence), but it always exist, or there is no point of radiation transport.

$$u(\vec{r}) = \frac{dE(\vec{r})}{dV}$$

We can separate this energy density according to wavelength



Spectral energy density is basically a distribution function of the energy density over wavelengths. It cannot be directly measured.

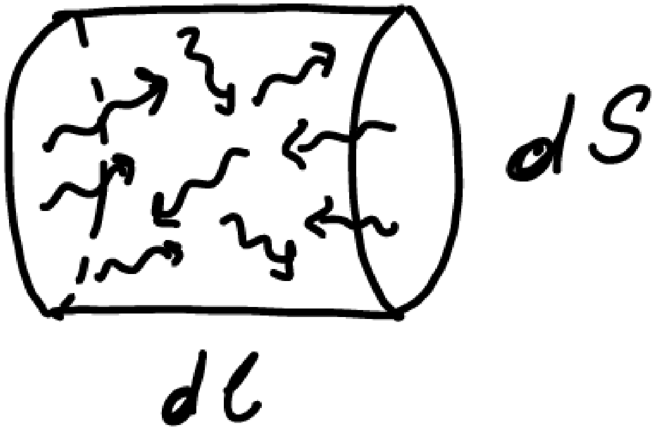
$$u_\lambda = u(\lambda) = \frac{du}{d\lambda}$$

We measure this

$$u_\lambda = \frac{dE}{dV\,d\lambda}$$

But not this

Energy passing through a surface, in a unit of time, in a unit of wavelength = **spectral flux density**



Convince yourself that this is zero if the radiation is *isotropic*

$$F_\lambda = \frac{dE}{dS\, dt\, d\lambda}$$

$$F_\lambda^+ = \frac{dE^+}{dS\, dt\, d\lambda}$$

Convince yourself that, for isotropic radiation:

$$F_\lambda^+ = \frac{c}{2} u_\lambda$$

Solution:

We are looking only for the outgoing radiation. If it's isotropic, half goes out , so:

$$F_\lambda^+ = \frac{1}{2}\frac{dE}{dS\,dt\,d\lambda} = \frac{1}{2}\frac{dE}{dV\,d\lambda}\frac{dl}{dt} = \frac{c}{2}u_\lambda$$

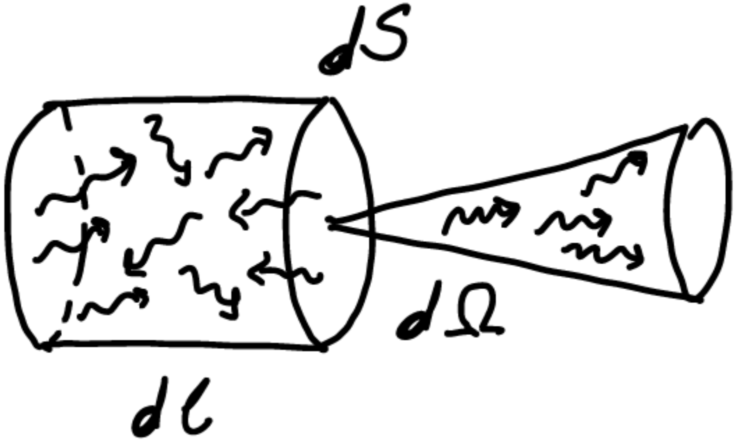
We can later define unsigned Flux where we add the absolute values of fluxes flowing in and out of the surface. For isotropic radiation:

$$F_\lambda^{\text{unsigned}} = c\,u_\lambda$$

What did we describe so far

- Specific flux tells us about the spatial, spectral and temporal distribution of the energy flow.
- There is one more thing to define – ***Direction***
- There is obviously a difference between a laser that emits the light in a very narrow direction and, say, a light bulb that basically emits everywhere in space
- This is the final dependency we are going to include.

Specific monochromatic intensity



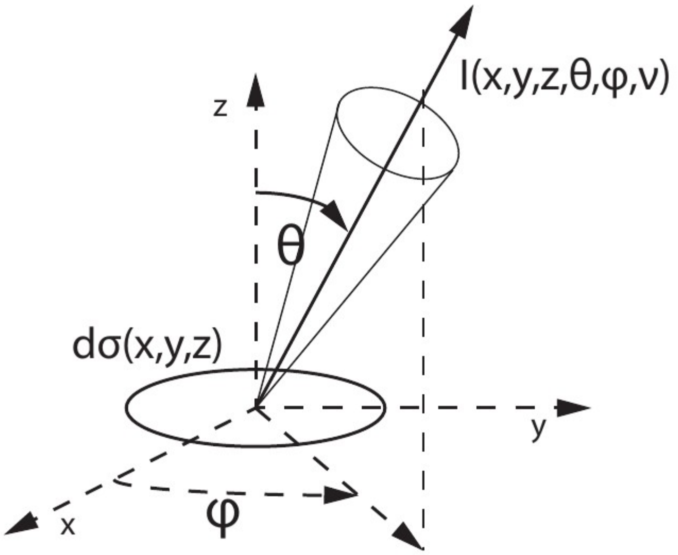
Now we are focusing only on the part of the specific monochromatic flux that are being transported (it can also be emitted, received, depending on the context) within this solid angle (cone).

$$I_\lambda = \frac{dE}{dS\,dt\,d\Omega\,d\lambda} \left[\frac{W}{\mathrm{m}^2\,\mathrm{srad}\,\text{Å}}\right]$$

Or to finally show you a nicer figure...

In general, we also want to allow for inclined directions, and we end up with the following expression for the intensity:

$$I_\lambda = \frac{dE}{dS\ \cos\theta\, dt\, d\Omega\, d\lambda} \left[\frac{W}{\mathrm{m^2\, srad\, \AA}}\right]$$



It should be clear now why the intensity depends on 7 (!) “coordinates.”

This also explains why radiative transfer is so scary. The number of degrees of freedom is much higher than, for say, density.

Speaking of which, is intensity vector or scalar (3 mins)?

Answer: It is a **scalar!**

- It can be confusing that something that apparently has a “direction” is a scalar
- But remember that intensity is **defined in every direction**
- One way to think about this is to say that photons can pass through each other while “chunks of material” that we model in HD equations cannot.
- I actually only completely understood this when I started writing intro for my PhD thesis.

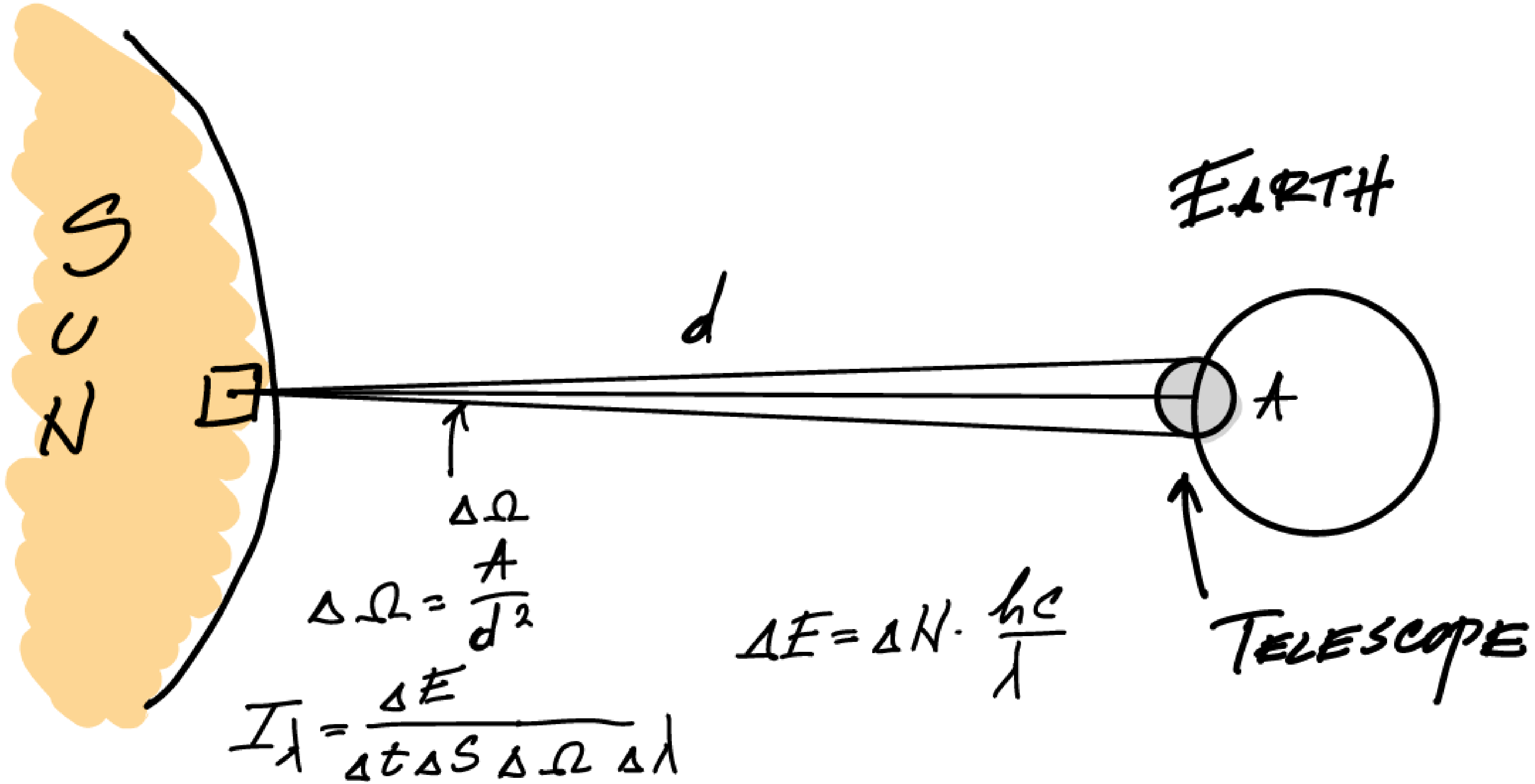
Let’s solve a problem related to the intensity

- Specific monochromatic intensity perpendicular to a square surface on the surface of the sum (100 km x 100 km) at wavelength of 630 nm is 2×10^{13} in SI units.

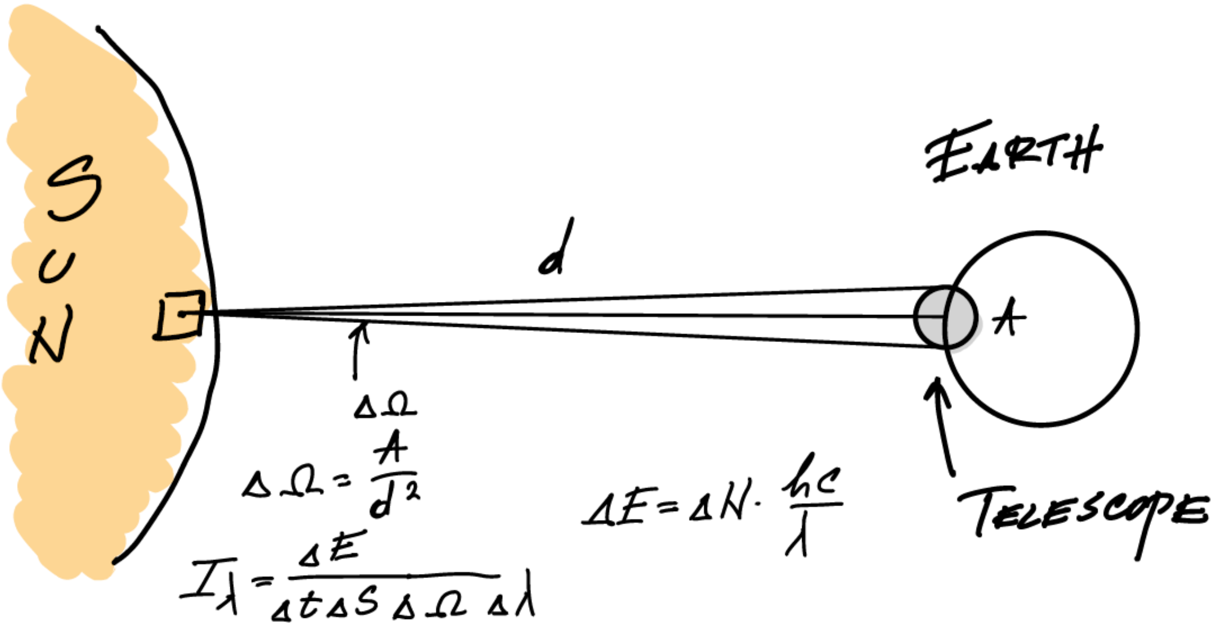
- How many photons per second in a 0.01 nm wide band are received by a telescope at Earth with effective collecting surface of $1 m^2$?

- Take 5 minutes to sketch and discuss and then we can take a look at the solution together.
  (Of course, I don’t care about the numbers, but it would be good if they made sense!)

Does this help a little bit?



So, finally:



$$\Delta E = I_\lambda \times \Delta\lambda \times \Delta\Omega \times \Delta t \times \Delta S$$

Anyone wants to guess the actual number?

$$\Delta N = \frac{630 \times 10^{-9}\,\mathrm{m}}{6.626 \times 10^{-34}\mathrm{Js} \times 3 \times 10^{8}\mathrm{m/s}} \times 2 \times 10^{13} \frac{\mathrm{J}}{\mathrm{m}^2\,\mathrm{s}\,\mathrm{m}}$$

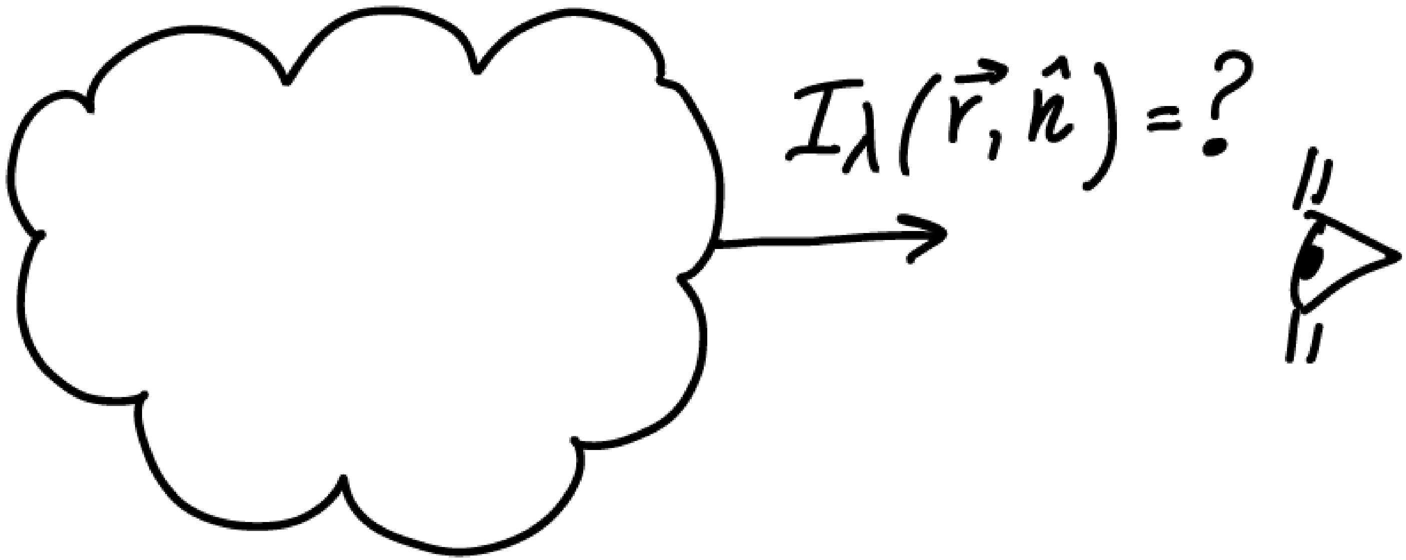
$$0.01 \times 10^{-9}\mathrm{m} \times 1\mathrm{s} \times \frac{1\,\mathrm{m}^2}{(149 \times 10^{9}\,\mathrm{m})^2} \times 10^{10}\mathrm{m}^2$$

Nice numbers, but why?

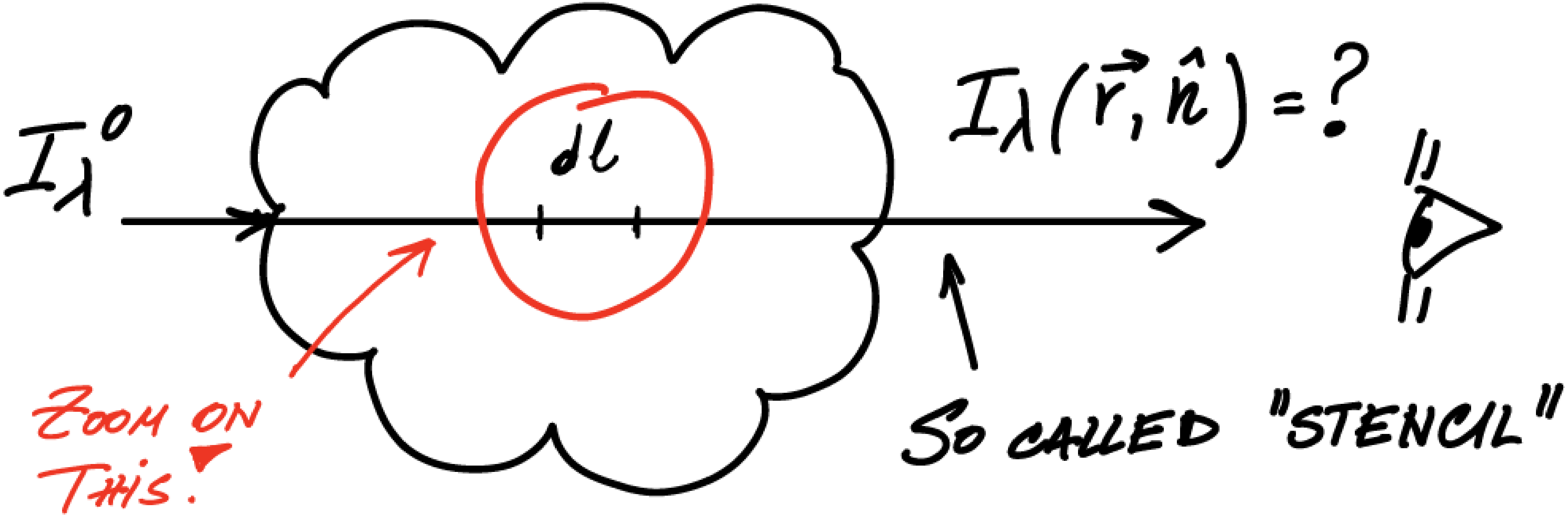
- We will see that in the modeling we want to use intensity specifically because it completely describes the radiation field
- Also, interestingly, if there are no sources or sinks of the radiation, intensity does not change along the given path
- Consequently, there is a handy *reciprocity theorem* that states that the intensity we receive is exactly the same as intensity that leaves the surface of the Sun (should be evident if you replace the positions of two surfaces)

Modeling the intensity

- Let's say that we want to calculate intensity in a given point, direction and wavelength (remember, everything is time – independent). For example, emerging from a surface of given object.



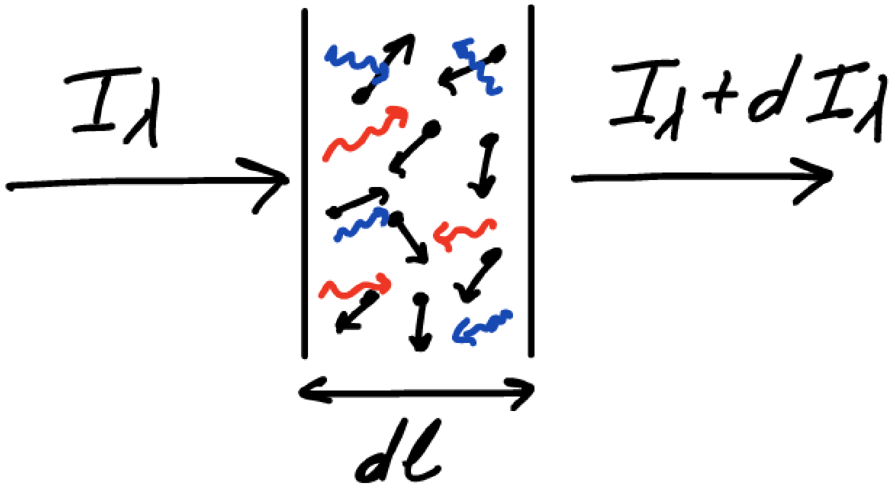
This is how we get to the radiative transfer equation



Radiative Transfer Equation

- A differential equation that describes how the intensity changes over a given path (ray, pencil, direction):

Inside of this infinitesimal "slab", we have absorption and emission processes



This is the simplest way to express radiative transfer equation in the "along-the-ray" form: It basically has 1D spatial dependence and, since we fixed the direction, no angular dependence. Wavelength dependence is, of course, still there.

$$\frac{dI_\lambda}{dl} = j_\lambda - \chi_\lambda I_\lambda$$

Understanding coefficient of emission

If we focus on the “emitted” part of the intensity, we would get:

$$\frac{dI_\lambda^{\mathrm{emitted}}}{dl} = j_\lambda$$

So, the emissivity quantifies amount of intensity added per unit length.

- Now, we can unpack it further:

$$j_\lambda = \frac{dE^{\mathrm{emitted}}}{dS\,dl\,dt\,d\Omega\,d\lambda} = \frac{dn^{\mathrm{emitted}}hc/\lambda}{dt\,d\Omega\,d\lambda}$$

Which is why sometimes you will find it called “volume emission coefficient”

The absorption coefficient

- We can have a very similar story here:

$$\frac{dI_\lambda^{\mathrm{abs}}}{dl} = -\chi_\lambda I_\lambda$$

- Note the different form: the absorbed intensity is proportional to the (input) intensity. Does that make sense to you? Any analogies? (2 mins)

The absorption coefficient (opacity)

- We can have a very similar story here:

$$\frac{dI_\lambda^{\mathrm{abs}}}{dl} = -\chi_\lambda I_\lambda$$

- Note the different form: the absorbed intensity is proportional to the (input) intensity. Does that make sense to you? Any analogies? (2 mins)
- The intensity should have units of inverse length. What if we relate it somehow to number density of absorption events (absorbers?)

$$\chi_\lambda[m^{-1}] = n^{\mathrm{absorbers}}[\mathrm{m}^{-3}] \times \sigma[\mathrm{m}^2]$$

- This is more “classical” than the previous argument, but it very intuitively tells us what opacity depends on!

Mean free path

- More opacity, shorter path photons can traverse before being absorbed (or, generally, removed from the ray)
- This means, more opaque the medium is more “trapped” the photons are
- This helps establish “local” thermodynamic equilibrium: photons are trapped in a thin layer and they establish equilibrium with the matter
- Keep in mind that the mean free path is also wavelength dependent

$$\bar{l}_\lambda = \frac{1}{\chi_\lambda}$$

Again, we will not go into the mechanics of emission and absorption coefficients.
That happens a bit later (Thursday ++) :-)

Solving radiative transfer equation

- This is very reasonable and understandable form:

$$\frac{dI_\lambda}{dl} = j_\lambda - \chi_\lambda I_\lambda$$

- Let's make it worse and less understandable and divide both sides with opacity (or if you want, minus opacity):

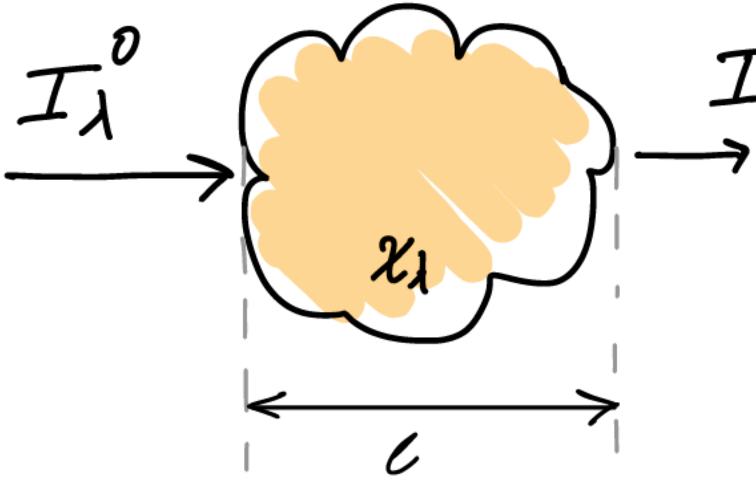
$$\frac{dI_\lambda}{\chi_\lambda dl} = I_\lambda - \frac{j_\lambda}{\chi_\lambda}$$

- And let's see what we get here

Optical depth

- What we did is we changed variable from geometrical path, that is insensitive to optical properties of the medium to a dimensionless quantity that depends on the opacity of the medium.

- Example, a homogeneous cloud that only absorbs:



$$\frac{dI_\lambda}{dl} = -\chi_\lambda I_\lambda$$

$$\frac{dI_\lambda}{I_\lambda} = -\chi_\lambda \cdot dl$$

$$\int_{I_\lambda^0}^{I_\lambda^+} \frac{1}{I_\lambda} dI_\lambda = \int_0^l -\chi_\lambda dl$$

$$\ln I_\lambda^+ - \ln I_\lambda^0 = \underbrace{-\chi_\lambda l}_{\tau_\lambda} \rightarrow I_\lambda^+ = I_\lambda^0 \cdot e^{-\tau_\lambda}$$

Note that, by looking at original and emergent intensity, we cannot tall whether it's a small dense cloud or a large tenuous cloud

Source function

- We also define another new quantity, **the Source function**, as the ratio between emission and absorption coefficients.

$$S_\lambda = \frac{j_\lambda}{\chi_\lambda}$$

- Radiative Transfer equation becomes:

$$\frac{dI_\lambda}{d\tau_\lambda} = S_\lambda - I_\lambda$$

- We have a new form. Independent coordinate depends on the wavelength. But the solution will be very simple

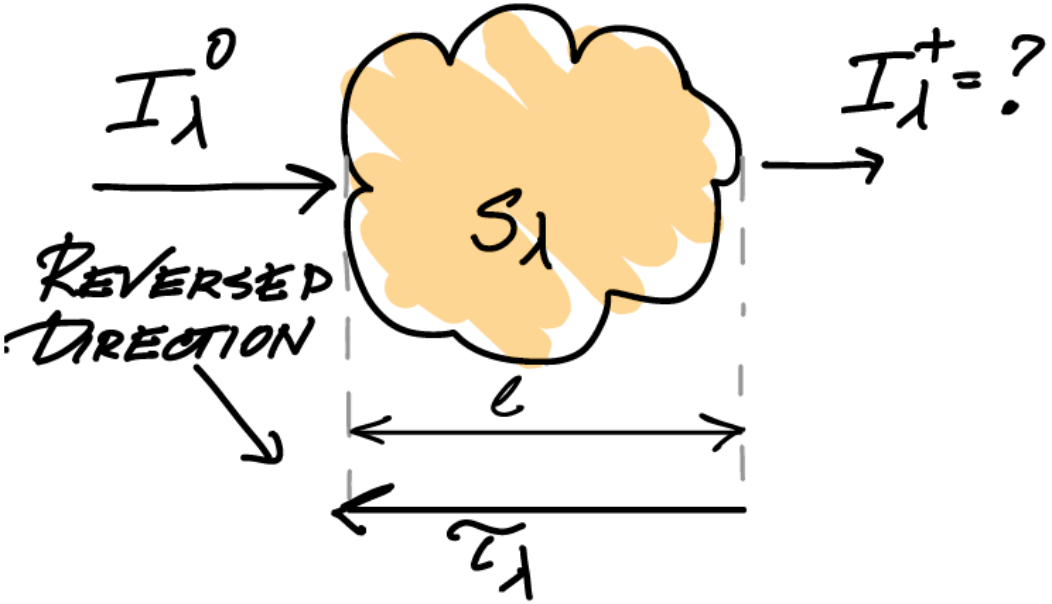
Some notes on the source function

$$\frac{dI_\lambda}{d\tau_\lambda} = S_\lambda - I_\lambda \qquad S_\lambda = \frac{j_\lambda}{\chi_\lambda}$$

- Source function has the same units as the intensity.
- One way to think about it is: **intensity added to the ray per unit optical path**
- People often say that optical depth is a natural coordinate for radiative transfer. It is because from the outside we cannot tell the geometrical scale (think about earlier argument about the cloud)

Formal solution

- We will often want to reverse the direction of optical depth so that 0 is at the top. This leads to the form you are maybe most familiar with:



$$\frac{dI_\lambda}{d\tau_\lambda} = I_\lambda - S_\lambda$$

With a solution:

$$I_\lambda^+ = I_\lambda^0 e^{-\tau_\lambda} + \int_0^{\tau_\lambda} S(t) e^{-t} dt$$

Formal solution

- What do we need to be able to solve this integral?

$$I_\lambda^+ = I_\lambda^0 e^{-\tau_\lambda} + \int_0^{\tau_\lambda} S(t) e^{-t} dt$$

- Incident intensity (boundary condition), we need run of S on the optical depth and for that we need the opacity.
- It is called formal solution because it assumes we can explicitly calculate opacity and emissivity (emission coefficient). This is not always true.
- What are some things you can conclude from the formal solution? (3 mins)

Some things that came to my mind

$$I_\lambda^+ = I_\lambda^0 e^{-\tau_\lambda} + \int_0^{\tau_\lambda} S(t) e^{-t} dt$$

- More optically thick (deep) the object is, less the incident intensity matters.
- The source function closer to the surface (t=0), matters more.
- Actually, Source function in great depths matters very little.
- It's all exponentials, so things really fall of quick for monochromatic optical depth above "a few"
- Everything is wavelength dependent and wavelengths are de-coupled, which means we can solve this wavelength-by-wavelength.

Simplified solutions – constant source function

- Let's assume the cloud is indeed homogeneous and the source function is constant everywhere

$$I_\lambda^+ = I_\lambda^0 e^{-\tau_\lambda} + \int_0^{\tau_\lambda} S(t) e^{-t} dt$$

$$I_\lambda^+ = I_\lambda^0 e^{-\tau_\lambda} + S_\lambda (1 - e^{-\tau_\lambda})$$

- Emergent intensity is weighted contribution of the incoming intensity and the source function. The weighting depends on the optical depth of the object.

Simplified solutions – linear source function

- Let's assume that source function linearly increases with optical depth
- In addition, it is common to assume that the optical depth is very large (infinite). We will use this to model solar atmosphere later

$$I_\lambda^+ = I_\lambda^0 e^{-\tau_\lambda} + \int_0^{\tau_\lambda} (a + bt) e^{-t} dt$$

$$I_\lambda^+ = a + b = S(\tau_\lambda = 1)$$

- The emergent intensity is approximately equal to Source function at the optical depth unity *at corresponding wavelength:* ***Eddington – Barbier relationship.***

Wavelength scaling

- Believe it or not, the optical depth is more dramatically scaling with wavelength than the source function
- The reason for that is that optical depth depends on the opacity (absorption coefficient) only, while the source function depends on the ratio, so wavelength dependence somewhat "cancels"
(next class we will see that this is actually super true)
- Let's then assume that the optical depth has some "profile" in wavelength, while the source function is constant. And move to exercises

Ok, jupyter notebook time!