Hale COLLAGE problem set 4 (SOLUTIONS)

We represent the non-thermal component of the electron distribution by the angle-averaged distribution function $f(v)$ defined so that the number density and energy density of non-thermal electrons are

$$n_{nt} \;=\; \int_0^\infty f(v)\,dv \quad , \quad \varepsilon_{nt} \;=\; \tfrac{1}{2}m_e\int_0^\infty v^2 f(v)\,dv \;=\; \tfrac{3}{2}m_e\,n_{nt}\,v_{nt}^2 \quad , \tag{1}$$

where v_{nt} is a characteristic velocity for the non-thermal electrons. The non-thermal distribution function evolves according to the Fokker-Planck equation like that on slide 32 of lecture 18. We will write this as

$$\frac{\partial f}{\partial t} \;=\; \left(\frac{\partial f}{\partial t}\right)_c \;=\; \frac{\partial}{\partial v}\left[\frac{K(v^2-2v_{th}^2)}{v^4}\,f \;+\; \left(\frac{Kv_{th}^2}{v^3}+D^{(\mathrm{turb})}\right)\frac{\partial f}{\partial v}\right] \quad , \tag{2}$$

where $K = 4\pi e^4 n_{th}\Lambda/m_e^2$ is a constant, and Λ is the Coulomb logarithm. The density of non-thermal electrons $n_{nt} \ll n_{th}$, the number density of thermal electrons with which the non-thermal electrons collide. It is only this larger density which appears in the collision operator through the constant K. The thermal velocity of the target electrons, $v_{th}^2 = k_bT/m_e$, is also part of the collision operator.

a. Show that when there is no turbulent contribution, i.e. $D^{(\mathrm{turb})} = 0$, a Maxwellian with $v_{nt} = v_{th}$ is a steady solution. Be aware that this is a Maxwellian version of $f(v)$, not $f(\mathbf{v})$ — the difference is explained on slide 29 of lecture 17.

b. Now include an arbitrary turbulent diffusion $D^{(\mathrm{turb})}(v) \neq 0$ and find an expression for the change of non-thermal energy density $\partial\varepsilon_{\mathrm{nt}}/\partial t$. Integrate by parts to obtain an expression involving only moments of f and possibly surface terms. When doing so assume that

$$f(v) \;\rightarrow\; C\,v^2 \quad , \quad v \;\rightarrow\; 0 \quad , \tag{3}$$

for some constant C, and that $f(v)$ vanishes more rapidly than any inverse power of v as $v \rightarrow \infty$. Assume also that $v^2D^{(\mathrm{turb})} \rightarrow 0$ as $v \rightarrow 0$. Surface terms in your final expression will include C. The full expression should contain contributions from collisions, proportional to K, and from turbulence. The latter term should vanish if $D^{(\mathrm{turb})}(v) \propto v^{-1}$.

c. Take $f(v)$ to be a Maxwellian with thermal speed v_{nt}, and show that the collisional contribution to the energy change, found in part b., is proportional to $v_{th}^2 - v_{nt}^2$. If you begin with $v_{nt} > v_{th}$ will collisions increase or decrease the non-thermal energy ε_{nt}?. Explain in words how *elastic* collisions can result in *any* change to energy ε_{nt}.

d. Now take the turbulent diffusion to be of the form which will not change the energy of the non-thermal particles (so the turbulence itself will neither gain nor lose energy to the particles)

$$D^{(\mathrm{turb})} \;=\; \frac{G}{v} \quad , \tag{4}$$

where G is a constant. (We showed in lecture 19 that this constant is related to the energy density of the turbulence as $G = (2\pi e/m_e)^2\varepsilon_{\mathrm{turb}}/\bar{k}$, where $\bar{k}$ is a wavenumber characteristic of the turbulent spectrum.) Show that the Fokker-Planck equation is exactly solved by a steady state distribution of the form

$$f(v) \;=\; C\,v^2\left(1+\beta v^2\right)^{-(\delta+1)} \quad , \tag{5}$$

where C, β and δ are all constants. This combines a Maxwellian-like core (i.e. $\beta v^2 < 1$) and power-law tail into a single seamless function The exponent is named δ so that the non-thermal energy flux $F(E) \sim f(v) \to E^{-\delta}$ matching the traditional usage. Write down the values of the constants δ and β, which make expression (5) a steady solution, explicitly in terms of K, G and v_{th}.

e. For the solution found in part d. show that it assumes the form of a Maxwellian in the limit $G \to 0$. Show only that the functional form is correct in that limit, disregarding the normalization constant C. But show also that the Maxwellian obtained in this limit has a width v_{th}.

f. For the solution found in part d. perform integrals in eq. (1) to obtain expressions for C, in terms of n_{nt}, v_{th} and δ. Then find the non-thermal velocity v_{nt}. Show that v_{nt} approaches the expected limit when $\delta \to \infty$. Which values of δ yield a finite value of v_{nt}? The integrals may be performed using

$$I_p \;=\; \int_0^\infty v^{2p}\,(1+\beta v^2)^{-(\delta+1)}\,dv \;=\; \frac{\Gamma\left(\delta - p + \frac{1}{2}\right)\Gamma\left(p+\frac{1}{2}\right)}{2\beta^{(p+1/2)}\Gamma(\delta+1)} \quad , \tag{6}$$

where $\Gamma(x)$ is the Γ function defined so that $\Gamma(x+1) = x\Gamma(x)$.

g. Since eq. (5) is a steady equilibrium, the effects of turbulence and collision must cancel at each point in the distribution. The change due to turbulence alone

$$\left(\frac{\partial f}{\partial t}\right)_{\rm turb} \;=\; \frac{\partial}{\partial v}\left(\frac{G}{v}\frac{\partial f}{\partial v}\right) \quad , \tag{7}$$

will either add or subtract particles at a given velocity — collisions will do the opposite. When the distribution has achieved its steady state, over what range of velocities does the turbulence *add* particles? (Naturally it will *remove* the same number from outside this region, since wave-particle interactions will neither create nor destroy particles.)

SOLUTION:

a. A Maxwellian may be written, up to a pre-factor,

$$f(v) \;=\; v^2 \,\exp\left(-\frac{v^2}{2v_{th}^2}\right) \quad , \tag{8}$$

where we have used the same thermal velocity as the thermal electron distribution. Taking its derivative gives

$$\frac{\partial f}{\partial v} \;=\; \left(2v \;-\; \frac{v^3}{v_{th}^2}\right)\exp\left(-\frac{v^2}{2v_{th}^2}\right) \;=\; \frac{1}{v\,v_{th}^2}\left(2v_{th}^2 \;-\; v^2\right)\, f(v) \quad . \tag{9}$$

The factor appearing second inside the square brackets of in eq. (2) is therefore

$$\frac{Kv_{th}^2}{v^3}\frac{\partial f}{\partial v} \;=\; -\,\frac{K(v^2 \;-\; 2v_{th}^2)}{v^4}\, f(v) \quad . \tag{10}$$

This clearly cancels the first term in brackets, demonstrating that the Maxwellian, eq. (8) is a steady-state solution.

b. Taking the time derivative of expression (1) yields

$$\begin{aligned}
\frac{\partial \varepsilon_{nt}}{\partial t} \;&=\; \tfrac{1}{2}m_e \int_0^\infty v^2 \frac{\partial f}{\partial t}\,dv \;=\; \tfrac{1}{2}m_e \int_0^\infty v^2 \frac{\partial}{\partial v}\left[\frac{K(v^2-2v_{th}^2)}{v^4}\,f + \left(\frac{Kv_{th}^2}{v^3} + D^{(\mathrm{turb})}\right)\frac{\partial f}{\partial v}\right] dv \\
&=\; -m_e \int_0^\infty K(v^2-2v_{th}^2)\,\frac{f}{v^3}\,dv \;-\; m_e \int_0^\infty \left(\frac{Kv_{th}^2}{v^2} + v\,D^{(\mathrm{turb})}\right)\frac{\partial f}{\partial v}\,dv \quad ,
\end{aligned} \tag{11}$$

after integration by parts. The boundary term in this integration vanishes

$$\begin{aligned}
\text{bndry term} \;&=\; \tfrac{1}{2}m_e \left[\frac{K(v^2-2v_{th}^2)}{v^2}\,f + \left(\frac{Kv_{th}^2}{v} + v\,D^{(\mathrm{turb})}\right)\frac{\partial f}{\partial v}\right]\Bigg|_{v=0}^{\infty} \quad , \\
&=\; m_e\,K\,C\,v_{th}^2 \;-\; m_e\,K\,C\,v_{th}^2 \;=\; 0 \quad ,
\end{aligned} \tag{12}$$

after using asymptotic form given in eq. (3), discarding the upper limit ($v \to \infty$), and the turbulent piece. The second term may be integrated by parts once more to yield

$$\begin{aligned}
\frac{\partial \varepsilon_{nt}}{\partial t} \;&=\; -m_e \int_0^\infty K(v^2-2v_{th}^2)\,\frac{f}{v^3}\,dv \;-\; m_e\left(\frac{Kv_{th}^2}{v^2} + v\,D^{(\mathrm{turb})}\,f\right)\Bigg|_{v=0}^{\infty} \\
&\quad + m_e \int_0^\infty \left[-2\frac{Kv_{th}^2}{v^3} + \frac{\partial(v\,D^{(\mathrm{turb})})}{\partial v}\right] f\,dv \\
&=\; m_e\,K\left[C\,v_{th}^2 \;-\; \int_0^\infty \frac{f}{v}\,dv\right] \;+\; m_e \int_0^\infty \frac{\partial\left[v\,D^{(\mathrm{turb})}\right]}{\partial v}\,f\,dv \quad ,
\end{aligned} \tag{13}$$

where the boundary term yields the factor $\propto C$ via eq. (3). This clearly contains two contributions. The first involves collisional terms, and the second the turbulent piece. It is also clear that if $D^{(\mathrm{turb})}(v) \propto v^{-1}$ the second contribution will vanish.

So the solution is

$$\boxed{\frac{\partial \varepsilon_{nt}}{\partial t} \;=\; m_e\, K\left[C\, v_{th}^2 \;-\; \int_0^\infty \frac{f}{v}\, dv\right] \;+\; m_e \int_0^\infty \frac{\partial\left[v\, D^{(\mathrm{turb})}\right]}{\partial v}\, f\, dv \quad . \qquad (14)}$$

c. The Maxwellian form for $f(v)$

$$f(v) \;=\; \sqrt{\frac{2}{\pi}}\, \frac{n_{nt}}{v_{nt}^3}\, v^2 \exp\left(-\frac{v^2}{2v_{nt}^2}\right) \quad , \tag{15}$$

satisfies both integrals from (1). It clearly has

$$C \;=\; \sqrt{\frac{2}{\pi}}\, \frac{n_{nt}}{v_{nt}^3} \quad . \tag{16}$$

The integral we need to perform is

$$\int_0^\infty \frac{f}{v}\, dv \;=\; \sqrt{\frac{2}{\pi}}\, \frac{n_{nt}}{v_{nt}^3} \int_0^\infty v \exp\left(-\frac{v^2}{2v_{nt}^2}\right) dv \;=\; \sqrt{\frac{2}{\pi}}\, \frac{n_{nt}}{v_{nt}} \tag{17}$$

Placing this into eq. (14) yields the change rate

$$\boxed{\frac{\partial \varepsilon_{nt}}{\partial t} \;=\; m_e\, K\left[C\, v_{th}^2 \;-\; \int_0^\infty \frac{f}{v}\, dv\right] \;=\; \sqrt{\frac{2}{\pi}}\, \frac{m_e K n_{nt}}{v_{nt}^3}\left[v_{th}^2 \;-\; v_{nt}^2\right] \quad , \qquad (18)}$$

which takes the form predicted. If $v_{th} < v_{nt}$, then the non-thermal energy will decrease; it will decrease v_{nt} until it matches v_{th}.

d. Proposing the form

$$f(v) \;=\; v^2\left(1 + \beta v^2\right)^{-(\delta+1)} \quad . \tag{19}$$

and taking its derivative gives

$$\frac{\partial f}{\partial v} \;=\; \left[2v \;-\; \frac{2(\delta+1)\beta v^3}{1+\beta v^2}\right]\left(1+\beta v^2\right)^{-(\delta+1)} \;=\; \frac{2}{v}\left(\frac{1-\delta\beta v^2}{1+\beta v^2}\right) f(v) \quad . \tag{20}$$

The factor appearing second inside the square brackets of in eq. (2) is therefore

$$\left(\frac{K v_{th}^2}{v^3} + \frac{G}{v}\right)\frac{\partial f}{\partial v} \;=\; \frac{2K}{v^4}\left(v_{th}^2 + \frac{G}{K} v^2\right)\left(\frac{1-\delta\beta v^2}{1+\beta v^2}\right) f(v) \tag{21}$$

Finally, the entire factor in square brackets becomes

$$\begin{aligned} S \;&=\; \left[\frac{K(v^2 - 2v_{th}^2)}{v^4}\, f \;+\; \left(\frac{K v_{th}^2}{v^3} + \frac{G}{v}\right)\frac{\partial f}{\partial v}\right] \\ &=\; \frac{K}{v^4(1+\beta v^2)}\left[(v^2 - 2v_{th}^2)(1+\beta v^2) + 2\left(v_{th}^2 + \frac{G}{K}v^2\right)(1-\delta\beta v^2)\right] f(v) \\ &=\; \frac{CK}{v^2(1+\beta v^2)^{\delta+2}}\left[(v^2 - 2v_{th}^2)(1+\beta v^2) + 2\left(v_{th}^2 + \frac{G}{K}v^2\right)(1-\delta\beta v^2)\right] \quad , \end{aligned} \tag{22}$$

after introducing eq. (5) for $f(v)$. To be a steady solution it is necessary that $\partial S/\partial v = 0$, so S must be a constant. The term in square brackets of eq. (22) is a quadratic polynomial in v^2. In order for S to be a constant other than zero, the denominator would also need to be quadratic in v^2 — the same one. This in turn requires $\delta = -1$ so $f(v) = Cv^2$ — this is preposterous. It is therefore evident that S must not just be constant, but it *must be zero.* This amounts to requiring the quadratic in square brackets of (22) to vanish; it must vanish term by term. The leading order term, proportional to $(v^2)^2 = v^4$,

$$\left(\beta \;-\; 2\frac{G}{K}\delta\beta\right)\, v^4 \quad , \tag{23}$$

will vanish only if

$$\boxed{\delta \;=\; \frac{K}{2G} \quad .} \tag{24}$$

The term linear in v^2 is

$$\left(1 \;-\; 2v_{th}^2\beta \;+\; 2\frac{G}{K} \;-\; 2\delta\beta v_{th}^2\right)\, v^2 \;=\; \left[1 \;+\; \frac{1}{\delta} \;-\; 2v_{th}^2\beta(1+\delta)\right] \quad , \tag{25}$$

will vanish provided

$$\boxed{\beta \;=\; \frac{1}{2\delta v_{th}^2} \;=\; \frac{G}{K\, v_{th}^2} \quad .} \tag{26}$$

Finally, the constant term vanishes for any choice of δ and β. The steady-state solution therefore requires δ and β set according to (24) and (26).

e. In the limit $G \to 0$, expression (24) shows that $\delta \to \infty$. The steady distribution takes the form

$$f(v) \;=\; C\, v^2 \left(1 + \frac{v^2}{2\delta\, v_{th}^2}\right)^{-(\delta+1)} \quad , \tag{27}$$

after using (26) to substitute for β. We use the fact

$$\lim_{\delta\to\infty} \left(1 + \frac{v^2}{2\delta\, v_{th}^2}\right)^{-(\delta+1)} \;=\; \exp\left(-\frac{v^2}{2\, v_{th}^2}\right) \quad , \tag{28}$$

to see that $f(v)$ approaches a Maxellian in that limit. This fact can be establised taking the logarithm

$$\begin{aligned}
\ln\left[\lim_{\delta\to\infty} \left(1 + \frac{v^2}{2\delta\, v_{th}^2}\right)^{-(\delta+1)}\right] \;&=\; -\lim_{\delta\to\infty}\left[(\delta+1)\ln\left(1 + \frac{v^2}{2\delta\, v_{th}^2}\right)\right] \\
&=\; -\lim_{\delta\to\infty}\left[(\delta+1)\,\frac{v^2}{2\delta\, v_{th}^2}\right] \;=\; -\frac{v^2}{2\, v_{th}^2} \quad .
\end{aligned} \tag{29}$$

Thus the distribution does approach a Maxwellian in the limit of vanishing turbulence.

f. The zeroth moment of expression (5) is

$$
\begin{aligned}
n_{nt} &= C\int_0^\infty v^2(1+\beta v^2)^{-(\delta+1)}\,dv \;=\; C\,I_1 \;=\; \frac{C\,\Gamma\left(\delta-\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right),}{2\beta^{3/2}\Gamma(\delta+1)} \\
&= \frac{C\,\Gamma\left(\frac{1}{2}\right)\Gamma\left(\delta-\frac{1}{2}\right)}{4\beta^{3/2}\Gamma(\delta+1)} \;=\; \frac{C\,\sqrt{\pi}\,\Gamma\left(\delta-\frac{1}{2}\right)}{4\beta^{3/2}\Gamma(\delta+1)} \quad , \qquad (30)
\end{aligned}
$$

after using the value $\Gamma(1/2)=\sqrt{\pi}$. This yields the expression

$$
\boxed{C \;=\; \frac{4\,n_{nt}\,\beta^{3/2}}{\sqrt{\pi}}\frac{\Gamma(\delta+1)}{\Gamma\left(\delta-\frac{1}{2}\right)} \;=\; \sqrt{\frac{2}{\pi}}\,\frac{n_{nt}}{v_{th}^3}\,\frac{\Gamma(\delta+1)}{\delta^{3/2}\,\Gamma\left(\delta-\frac{1}{2}\right)} \quad . \qquad (31)}
$$

The second moment is

$$
\begin{aligned}
\varepsilon_{nt} &= \tfrac{1}{2}m_e\int_0^\infty v^2 f(v)\,dv \;=\; \frac{m_e C}{2}\int_0^\infty v^4(1+\beta v^2)^{-(\delta+1)}\,dv \;=\; \frac{m_e C}{2}\,I_2 \\
&= \frac{m_e C\,\Gamma\left(\delta-\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)}{4\beta^{5/2}\Gamma(\delta+1)} \;=\; \frac{3\,m_e\,\sqrt{\pi}\,C\,\Gamma\left(\delta-\frac{3}{2}\right)}{16\beta^{5/2}\Gamma(\delta+1)} \quad . \qquad (32)
\end{aligned}
$$

Substituting the first relationin eq. (31) yields

$$
\varepsilon_{nt} \;=\; \frac{3m_e\,n_{nt}}{4\beta}\,\frac{\Gamma\left(\delta-\frac{3}{2}\right)}{\Gamma\left(\delta-\frac{1}{2}\right)} \;=\; \tfrac{3}{2}\,m_e\,n_{nt}v_{th}^2\,\left(1\;-\;\frac{3}{2\delta}\right)^{-1} \quad . \qquad (33)
$$

Equating this to $(3/2)m_e n_{nt} v_{nt}^2$ yields the relation

$$
\boxed{v_{nt} \;=\; \frac{v_{th}}{\sqrt{1-3/2\delta}} \quad . \qquad (34)}
$$

In the limit $\delta\to\infty$, which occurs in the absence of turbulence (i.e. $G\to 0$) we find $v_{nt}\to v_{th}$, consistent with the results of part a.

The pre-factor given in eq. (31) also approaches the value appropriate for a Maxwellian in the limit $\delta\to\infty$. Showing this requires Stirling's approximation

$$
\ln\Gamma(x) \;\to\; x\ln(x) \;-\; x \;-\; \tfrac{1}{2}\ln\left(\frac{2\pi}{x}\right) \quad , \quad x\;\to\;\infty \quad . \qquad (35)
$$

This yields

$$
\begin{aligned}
\ln[\Gamma(x+\epsilon)] &\simeq (x+\epsilon)\ln(x+\epsilon) \;-\; x \;-\epsilon \;-\; \tfrac{1}{2}\ln\left(\frac{2\pi}{x+\epsilon}\right) \quad , \\
&\simeq (x+\epsilon)\left[\ln(x) \;+\; \ln\left(1+\frac{\epsilon}{x}\right)\right] \;-\; x \;-\epsilon \;-\; \tfrac{1}{2}\ln\left(\frac{2\pi}{x}\right) \;+\; \tfrac{1}{2}\ln\left(1+\frac{\epsilon}{x}\right) \quad , \\
&\simeq (x+\epsilon)\left[\ln(x) \;+\; \frac{\epsilon}{x}\right] \;-\; x \;-\epsilon \;-\; \tfrac{1}{2}\ln\left(\frac{2\pi}{x}\right) \;+\; \frac{\epsilon}{2x} \quad , \\
&\simeq \ln[\Gamma(x)] \;+\; \epsilon\,\ln(x) \;+\; \mathcal{O}\left(\frac{\epsilon}{x}\right) \quad . \qquad (36)
\end{aligned}
$$

This means that for large arguments[1]

$$\Gamma(x+\epsilon) \;\simeq\; x^{\epsilon}\,\Gamma(x) \quad , \tag{37}$$

and thus that

$$\frac{\Gamma(\delta+1)}{\Gamma\Big(\delta-\frac{1}{2}\Big)} \;\simeq\; \frac{\delta\,\Gamma(\delta)}{\delta^{-1/2}\Gamma(\delta)} \;\simeq\; \delta^{3/2} \quad . \tag{38}$$

Using this in eq. (31) yields

$$C \;\simeq\; \sqrt{\frac{2}{\pi}}\,\frac{n_{nt}}{v_{th}^3} \quad , \tag{39}$$

which matches (16), the coefficient for a Maxwellian.

As a final check on our solution we calculate $\partial\varepsilon_{nt}/\partial t$, which involves the integral

$$\int_0^\infty \frac{f(v)}{v}\,dv \;=\; C\,I_{1/2} \;=\; C\,\frac{\Gamma(\delta)\,\Gamma(1)}{2\beta\,\Gamma(\delta+1)} \;=\; \frac{C}{2\beta\delta} \;=\; C\,v_{th}^2 \quad , \tag{40}$$

after using (26). Using this in expression (14) shows that the non-thermal energy does not change when the distribution takes the assumed form.

g. Returning to the derivative in eq. (20) we find

$$\frac{G}{v}\frac{\partial f}{\partial v} \;=\; 2CG\,\frac{(1+\beta v^2)-(\delta+1)\beta v^2}{(1+\beta v^2)^{(\delta+2)}} \;=\; 2CG\,\frac{1-\delta\beta v^2}{(1+\beta v^2)^{(\delta+2)}} \quad . \tag{41}$$

The derivative of this expression yields

$$\begin{aligned}
\left(\frac{\partial f}{\partial t}\right)_{\rm turb} \;&=\; \frac{\partial}{\partial v}\left(\frac{G}{v}\frac{\partial f}{\partial v}\right) \;=\; -\,4CGv\,\frac{\delta\beta(1+\beta v^2)+(\delta+2)\beta(1-\delta\beta v^2)}{(1+\beta v^2)^{(\delta+3)}} \\
&=\; -4CGv\,\frac{2\beta(\delta+1)-\delta\beta^2(\delta+1)v^2}{(1+\beta v^2)^{(\delta+3)}} \\
&=\; 4CGv\beta(\delta+1)\frac{\delta\beta v^2-2}{(1+\beta v^2)^{(\delta+3)}} \;=\; 2CGv\beta(\delta+1)\frac{(v/v_{th})^2-4}{(1+\beta v^2)^{(\delta+3)}} \quad ,
\end{aligned} \tag{42}$$

after using eq. (26) to replace $\delta\beta = 1/2v_{th}^2$. From the final expression it is clear that turbulence adds particles over the range $v > 2v_{th}$, and removes them from $v < 2v_{th}$. This does not depend on the strength of the turbulence G.

Extra:

The foregoing has used asymptotic, high-energy ($v \gg v_{th}$) forms for diffusion terms. These are singular at $v \to 0$, while the real diffusion coefficients are not. The genuine low-energy ($v \ll v_{th}$) behavior is thereby replaced by behavior singular at $v = 0$. This singular behavior can be characterized by extending the distribution $f(v)$ into regions of negative v but taking $f(v) = 0$ there. Expression (41) approaches the finite value $2CG$ from above ($v \to 0+$), while

[1]This relation is exactly true for $\epsilon = 0$ (obviously) and $\epsilon = 1$ by a well-known property of $\Gamma(x)$. For the case $\epsilon = 2$, the same well-known property yields $\Gamma(x+2) = (x+1)x\Gamma(x)$, which only approximately matches $x^2\Gamma(x)$, provided $x \gg 1$.

it is zero when approached from below ($v \to 0-$). This discontinuity produces a Dirac delta function in the derivative in eq. (42)

$$\left(\frac{\partial f}{\partial t}\right)_{\rm turb} = \frac{\partial}{\partial v}\left(\frac{G}{v}\frac{\partial f}{\partial v}\right) = 2CG\left[v\beta(\delta+1)\frac{(v/v_{th})^2-4}{(1+\beta v^2)^{(\delta+3)}} + \delta(v)\right] . \qquad (43)$$

The singular term adds particles at $v = 0$ which the real turbulent diffusion would add over a range of low energies. Only when this new term is included does the change integrate to zero. The term from eq. (42), plotted in red along the top panel of fig. 1, clearly has a negative integral. This would, by itself, indicate that turbulence removes more particles than it adds. The negative integral is, however, offset by the positive contribution from the Dirac delta function, making the net contribution zero after all: turbulence neither creates nor destroys particles.

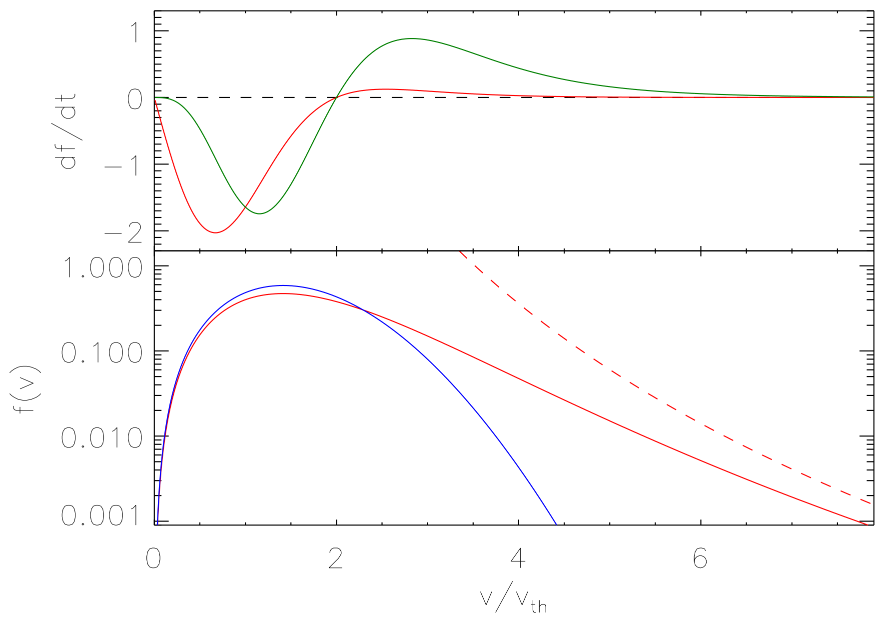

Figure 1: Plots of the steady distributions and their changes for the case $\delta = 4$. The bottom panel shows the Maxwellian (blue) and the function from (5), as a red curve. The red dashed curve shows the pure power law which $f(v)$ asymptotically approaches. The top panel shows $(\partial f/\partial t)_{\rm turb}$ from eq. (42) in red, and v^2/v_{th}^2 times this in green.

When the change $(\partial f/\partial t)_{\rm turb}$ is multiplied by v^2 (green curve in fig. 1) the contribution from $\delta(v)$ vanishes. The integral of this curve, proportional to the net energy change due to turbulence, vanishes all by itself. Thus the turbulent diffusion changes neither the number of particles nor the total energy density. It is evident that both curves cross zero at $v = 2v_{th}$. The turbulence removes energy from $v < 2v_{th}$ and adds it to $v > 2v_{th}$. In doing so, however, it does not change the net energy of the particles.

Collisions do exactly the opposite of all these things. They remove particles from $v \simeq 0$ and $v > 2v_{th}$, and add them to $0 < v < 2v_{th}$. They remove energy from $v > 2v_{th}$ and add it to $v < 2v_{th}$.