Hale COLLAGE problem set 3 (Due Fri. Mar. 24)

Flare models include some version of the radiative loss function $\Lambda(T)$. A monomial approximation

$$\Lambda(T) \simeq \Lambda_0\, T_6^{\alpha} \quad , \quad \Lambda_0 = 1.2\times 10^{-22}\,\mathrm{erg\,cm^3\,s^{-1}} \quad , \tag{1}$$

permits an analytic treatment. Here T_6 is the temperature expressed in units of Megakelvins, and α is some power-law index. Lecture 9 uses this form with $\alpha = -1/2$, which we argue to be a reasonable approximation to the actual function. Zero-dimensional models must use a version of this function, but it represents the average loss as a function of the average temperature. Once again a monomial approximation permits analytic treatment, but it is not clear that $\alpha = -1/2$ is still appropriate: the average of a function is not the same as the function of the average. We therefore re-derive some of our initial results, but using a general choice of α.

a. Using the form (1) for the radiative loss function, find an expression for the radiative cooling time, τ_{rad}. Express this in terms of T_6 and n_{10}, the electron density in units of $10^{10}\,\mathrm{cm}^{-3}$.

b. Use these results of part a. to find the relation between T_6 and n_{10} for which the radiative and conductive times are equal: $\tau_{\mathrm{rad}} = \tau_{\mathrm{cond}}$. Write n_{10} explicitly in terms of T_6 and L_9, the full length of the loop in units of 10^9 cm. This relations also approximates the condition for mechanical equilibrium. For which α are T and n_e correlated with one another, rather than anti-correlated?

c. Under the assumption of Antiochos & Sturrock (1978), evaporation decreases T and increases n_e, keeping constant their product n_eT, until $\tau_{\mathrm{rad}} = \tau_{\mathrm{cond}}$, at which point evaporation ends. The constant along which evaporation evolves is set by the total energy (per unit area) released in the flare, $\mathcal{E}$ ($\mathrm{erg\,cm^{-2}}$)

$$n_e\, T \;=\; \frac{\mathcal{E}}{3L\, k_b} \quad . \tag{2}$$

Use this to find the peak density, n_*, achieved at the end of evaporation, along with the corresponding temperature, T_*. Write down expressions for $n_{10,*}$ and $T_{6,*}$, involving only $\mathcal{E}_9$, L_9 and α.

d. Next assume that the radiative cooling process occurs in quasi-static equilibrium, so $\tau_{\mathrm{rad}} = \tau_{\mathrm{cond}}$, as found in part b. Klimchuk *et al* (2008) set the enthalpy equal to the difference between the conductive flux and the radiative loss form the transition region. They further set that loss to be 4 times the coronal radiative loss yields an energy equation[1]

$$\frac{3}{2}\frac{dp}{dt} \;=\; -\,5\, n_e^2\, \Lambda(T) \quad , \tag{3}$$

where $p = 2n_ek_bT$. Solve this to obtain an expression for $T_6(t)$, resembling slide 29 of lecture 9, but involving an arbitrary index α rather than the choice $\alpha = -1/2$. The solution should take the form

$$T(t) \;=\; T_*\Big(1+\omega\, t\Big)^{-\mu} \quad , \tag{4}$$

[1] The result of the assumptions is that conductive and enthalpy fluxes move energy around without changing it. All changes come from radiative losses which amount to $4+1=5$ times the coronal loss alone.

ion	λ Å	$G_\lambda^{(0)}$ $10^{-25}\,\mathrm{erg\,cm^3\,s^{-1}\,sr^{-1}}$	T_λ MK	σ_λ –
Fe XXI	128.7	1.64	11.51	0.30
Fe XVIII	93.9	1.43	6.91	0.44
Fe XIV	211.3	6.13	1.99	0.25
Fe IX	171.1	37.84	0.82	0.42

Table 1: Parameters defining the function $G_\lambda(T)$ for spectral lines in 4 different ions.

for constants of μ and ω you can write as explicit functions of $\mathcal{E}_9$, L_9 and α. Here we have taken $t = 0$ to be the beginning of radiative cooling — the time of peak density. Are there any values of α for which the solution vanishes in finite time?

e. Contribution functions from most optically thin spectral lines appear parabolic on log-log plots (see slides from lecture 11). They may therefore be approximated analytically as

$$G_\lambda(T) \;=\; G_\lambda^{(0)}\,\exp\left[-\frac{\ln^2(T/T_\lambda)}{\sigma_\lambda^2}\right] \quad , \tag{5}$$

where the constants $G_\lambda^{(0)}$, T_λ and σ_λ are constants for that particular spectral line. Values for 4 different spectral lines are given in table 1. Adopting this parameterization, show that during the radiative cooling phase the emissivity of a line, $\varepsilon_\lambda(t)$, will peak at a temperature

$$T_{\lambda,\mathrm{pk}} \;=\; \beta\,T_\lambda \quad , \tag{6}$$

for a factor β dependent only on α and σ_λ. Is β greater than or less than unity? Why? [The easiest approach to this result is to express $\ln(\varepsilon_\lambda)$ as a polynomial in $\ln(T)$, and find the maximum of *that*.]

f. As the loop cools it passes through peak emission at some time t_λ. The lifetime of its emission in this spectral line, $\Delta\tau_\lambda$, can be analytically defined though the relation

$$\frac{1}{\varepsilon_\lambda}\,\frac{d^2\varepsilon_\lambda}{dt^2}\bigg|_{t_\lambda} \;=\; -\,\frac{2}{\Delta\tau_\lambda^2} \quad . \tag{7}$$

Use the results above to show that

$$\Delta\tau_\lambda \;=\; \frac{\sigma_\lambda}{\omega\mu}\left(\frac{T_{\lambda,\mathrm{pk}}}{T_*}\right)^{-\nu} \quad , \tag{8}$$

for some index ν, depending on α, and ω and μ defined through eq. (4).

g. Now consider a particular flare with $L = 5\times10^9$ cm and $\mathcal{E} = 2\times10^{12}\,\mathrm{erg\,cm^{-2}}$ (the same values used to produce slide 34 of lecture 9). Return to the conventional choice of radiative loss function by setting $\alpha = -1/2$ (consistent with EBTEL). Consider each of the spectral lines in table 1 for which the loop *cools through the peak during its radiative phase*. For each of them find the time and duration of its peak emission, t_λ and $\Delta\tau_\lambda$.

SOLUTIONS:

a. The radiative cooling time is

$$\tau_{\rm rad} \;=\; \frac{\frac{3}{2}p}{n_e^2\Lambda(T)} \;=\; \frac{3\,n_e\,k_bT}{n_e^2\Lambda(T)} \;=\; \frac{3\times 1.38\,n_{10}\,T_6}{\Lambda_0\,10^{20}\,n_{10}^2T_6^{\alpha}} \;=\; (345\,{\rm s})\,T_6^{1-\alpha}\,n_{10}^{-1} \quad . \tag{9}$$

b. The conductive cooling time is still the same as in the lecture notes,

$$\tau_{\rm cond} \;=\; \frac{3\,n_e\,k_bT}{8\kappa_0 T^{7/2}/7L^2} \;=\; \frac{3\times 1.38\,n_{10}\,T_6}{1.14\times 10^{-3}\,T_6^{7/2}/L_9^2} \;=\; (3622\,{\rm s})\,T_6^{-5/2}\,n_{10}\,L_9^2 \quad . \tag{10}$$

Equating these yields the relation

$$\frac{\tau_{\rm rad}}{\tau_{\rm cond}} \;=\; \frac{345}{3622}\,T_6^{7/2-\alpha}\,n_{10}^{-2}\,L_9^{-2} \;=\; 1 \quad , \tag{11}$$

or

$$n_{10} \;=\; 0.31\,T_6^{(7-2\alpha)/4}\,L_9^{-1} \quad . \tag{12}$$

Temperature and density are correlated only for $\alpha < 7/2$. Otherwise they are anti-correlated. Such anti-correlation is a signature of a radiative instability.

c. The evaporation must occur along the curve

$$n_{10}\,T_6 \;=\; 10^{-16}\,\frac{\mathcal{E}}{3k_b\,L} \;=\; 0.24\,\mathcal{E}_9\,L_9^{-1} \quad . \tag{13}$$

Evaporation continues until $\tau_{\rm rad} = \tau_{\rm cond}$, which occurs when eq. (12) is satisfied. Introducing that yields

$$T_6^{(11-2\alpha)/4} \;=\; 0.77\,\mathcal{E}_9 \qquad \Longrightarrow \qquad T_{6,*} \;=\; (0.77\,\mathcal{E}_9)^{4/(11-2\alpha)} \quad . \tag{14}$$

Using this in eq. (12) yields the peak density

$$n_{10,*} \;=\; 0.31\,L_9^{-1}\,(0.77\,\mathcal{E}_{10})^{(7-2\alpha)/(11-2\alpha)} \quad . \tag{15}$$

As a check on these results we set $\alpha = -1/2$ to obtain

$$T_{6,*} \;=\; (0.77\,\mathcal{E}_9)^{1/3} \;=\; 0.93\,\mathcal{E}_9^{1/3} \quad , \quad n_{10,*} \;=\; 0.27\,L_9^{-1}\,\mathcal{E}_9^{2/3} \quad .$$

These expression agree with those on slide 32 of lecture 9, after using the fact that $\mathcal{E} = EL/V$.

d. Dividing the energy equation, eq. (3), by p yields

$$\frac{3}{2}\,\frac{1}{p}\frac{dp}{dt} \;=\; -\;5\frac{n_e^2}{p}\,\Lambda(T) \;=\; -\;\frac{3}{2}\,\frac{5}{\tau_{\rm rad}} \;=\; -\;\frac{3}{2}\,\frac{5}{\tau_{\rm cond}} \quad , \tag{16}$$

after introducing the cooling times which are equal. (Trick-of-the-trade: it is frequently easier to formulate ODEs in terms of logarithmic derivatives, as I am doing here. This cuts down on the number of constants needed.) If we now substitute $p = 2n_e k_b T$ we obtain

$$\frac{1}{p}\frac{dp}{dt} = \frac{1}{T}\frac{dT}{dt} + \frac{1}{n_e}\frac{dn_e}{dt} = -\frac{5}{\tau_{\rm cond}} = -(1.4\times 10^{-3}\,{\rm s}^{-1})\,T_6^{5/2}\,n_{10}^{-1}\,L_9^{-2} \quad . \tag{17}$$

We now use eq. (12) to re-write the logarithmic derivative of density

$$\frac{1}{n_e}\frac{dn_e}{dt} = \frac{1}{n_{10}}\frac{dn_{10}}{dt} = \frac{d\ln(n_{10})}{dt} = \frac{7-2\alpha}{4}\frac{d\ln(T_6)}{dt} = \frac{7-2\alpha}{4}\frac{1}{T}\frac{dT}{dt} \quad . \tag{18}$$

Placing this into eq. (17) yields a simple equation for T_6 alone

$$\begin{aligned}\frac{1}{p}\frac{dp}{dt} &= \left(1+\frac{7-2\alpha}{4}\right)\frac{1}{T}\frac{dT}{dt} = \left(\frac{11-2\alpha}{4}\right)\frac{1}{T_6}\frac{dT_6}{dt}\\ &= -(1.4\times 10^{-3}\,{\rm s}^{-1})\,T_6^{5/2}\left[0.31\,T_6^{(7-2\alpha)/4}\,L_9^{-1}\right]^{-1}L_9^{-2}\\ &= -\frac{(1.4\times 10^{-3}\,{\rm s}^{-1})}{0.31}\,T_6^{5/2-(7-2\alpha)/4}\,L_9^{-1} = -(4.5\times 10^{-3}\,{\rm s}^{-1})\,T_6^{(3+2\alpha)/4}\,L_9^{-1} \quad . \end{aligned} \tag{19}$$

This provides us with a simple ODE for T_6,

$$\frac{dT_6}{dt} = -(4.5\times 10^{-3}\,{\rm s}^{-1})\,\frac{4}{11-2\alpha}\,L_9^{-1}\,T_6^{1+(3+2\alpha)/4} \quad . \tag{20}$$

Dividing both sides by $T_6^{1+(3+2\alpha)/4}$ yields an equation

$$T_6^{-1-(3+2\alpha)/4}\frac{dT_6}{dt} = -\frac{4}{3+2\alpha}\frac{d}{dt}\left[T_6^{-(3+2\alpha)/4}\right] = -(4.5\times 10^{-3}\,{\rm s}^{-1})\,\frac{4}{11-2\alpha}\,L_9^{-1} \quad ,$$

which can be re-cast in the form

$$\frac{d}{dt}\left[T_6^{-(3+2\alpha)/4}\right] = \frac{3+2\alpha}{11-2\alpha}(4.5\times 10^{-3}\,{\rm s}^{-1})\,\frac{1}{L_9} \quad , \tag{21}$$

— very easy to solve. Dividing both sides by $T_{6,*}^{-(3+2\alpha)/4}$ yields an equation

$$\frac{d}{dt}\left(\frac{T_6}{T_{6,*}}\right)^{-(3+2\alpha)/4} = (4.5\times 10^{-3}\,{\rm s}^{-1})\frac{3+2\alpha}{11-2\alpha}\,\frac{T_{6,*}^{(3+2\alpha)/4}}{L_9} = \omega \quad , \tag{22}$$

where we have defined the right hand side to be ω, in anticipation of the form in (4). Using eq. (14) yields an explicit version

$$\omega = \left(\frac{1}{224\,{\rm s}}\right)\frac{3+2\alpha}{11-2\alpha}\,\frac{(0.77\mathcal{E}_9)^{(3+2\alpha)/(11-2\alpha)}}{L_9} \quad , \tag{23}$$

depending only on $\mathcal{E}_9$ and L_9, as well as on α.

Defining $t = 0$ to be the time at which radiative cooling begins, and thus $T_6 = T_{6,*}$, we have the solution

$$T(t) \;=\; T_* \Big(1+\omega\, t\Big)^{-\mu} \quad , \quad \mu \;=\; \frac{4}{3+2\alpha} \quad . \tag{24}$$

We can check this using the limit $\alpha = -1/2$, for which $\mu = 2$ and we find

$$T(t) \;=\; T_* \Big(1+\omega\, t\Big)^{-2} \quad ,$$

in agreement with slide 29 of lecture 9. That slide followed the approach of Cargill *et al* (1995) which neglected the enthalpy flux entirely. Doing so yields an inverse time-scale ω smaller than our treatment of enthalpy flux — smaller by a factor 2/5 in fact.

Since the loop is cooling, expression (24) must be a monotonically decreasing function of time. In the case

$$\alpha \;<\; -\frac{3}{2} \quad , \tag{25}$$

we find the exponent $\mu < 0$. Expression (23) reveals that for this same case, $\omega < 0$. Thus the temperature drops to zero at $t = 1/|\omega|$.

e. The emissivity of spectral line λ is

$$\varepsilon_\lambda \;=\; 4\pi\, n_e^2\, G_\lambda(T) \;=\; 4\pi\, n_e^2\, G_\lambda^{(0)}\, \exp\left[-\frac{\ln^2(T/T_\lambda)}{\sigma^2}\right] \quad . \tag{26}$$

Taking its natural log and using eq. (12) to eliminate n_e yields

$$\ln\Big(\varepsilon_\lambda\Big) \;=\; \frac{7-2\alpha}{2}\,\ln(T) \;-\; \frac{1}{\sigma^2}\Big[\ln(T) \;-\; \ln(T_\lambda)\Big]^2 \;+\; \text{const.} \;=\; f(T) \quad , \tag{27}$$

where the constants are not important for obtaining a maximum. Taking the derivative of this expression w.r.t. T, and setting it to zero, yields the equation

$$T\, f'(T) \;=\; \frac{7-2\alpha}{2} \;-\; \frac{2}{\sigma^2}\Big[\ln(T) \;-\; \ln(T_\lambda)\Big] \;=\; 0 \quad . \tag{28}$$

The temperature at peak emissivity is

$$T_{\lambda,\text{pk}} \;=\; T_\lambda\, \exp\Big[\tfrac{1}{4}\,\sigma^2(7-2\alpha)\Big] \;=\; \beta\, T_\lambda \quad , \tag{29}$$

Provided temperature and density both decrease during radiative cooling (i.e. $\alpha < 7/2$) the coefficient $\beta > 1$, and the emissivity will peak at a temperature slightly higher than the contribution function does. This occurs as the density is dropping just fast enough to offset the increase in contribution function as T decreases toward T_λ.

f. The time-dependent emissivity can be written

$$\varepsilon_\lambda \;=\; \exp\Big\{\, f[\,T(t)\,]\,\Big\} \quad , \tag{30}$$

where $f(T)$ is defined in eq. (27). Taking its first derivative yields

$$\frac{d\varepsilon_\lambda}{dt} \;=\; \frac{dT}{dt}\, f'(T)\, \exp\Big\{\, f[\,T(t)\,]\,\Big\} \;=\; \frac{dT}{dt}\, f'(T)\, \varepsilon_\lambda \quad , \tag{31}$$

after two applications of the chain rule. Naturally this vanishes at $T = T_{\lambda,\mathrm{pk}}$, since that is where $f'(T) = 0$. The only non-vanishing contribution to the second derivative will come from the term with $f''(T)$, since every other terms will involve at least one factor of $f'(T)$, which will vanish. The result is

$$\left.\frac{d^2\varepsilon_\lambda}{dt^2}\right|_{T_{\lambda,\mathrm{pk}}} \;=\; \left(\frac{dT}{dt}\right)^2 f''(T_{\lambda,\mathrm{pk}})\,\varepsilon_\lambda \quad . \tag{32}$$

Two derivatives of (27) yields

$$f''(T_{\lambda,\mathrm{pk}}) \;=\; -\,\frac{2}{T_{\lambda,\mathrm{pk}}^2\,\sigma^2} \quad . \tag{33}$$

Next we take the time derivative of eq. (24) to obtain

$$\frac{dT}{dt} \;=\; -\,\mu\,\omega\,T_*\,(1+\omega\,t)^{-\mu-1} \;=\; -\,\mu\,\omega\,T_*\,\left(\frac{T}{T_*}\right)^{(\mu+1)/\mu} \quad . \tag{34}$$

Combing these pieces yields

$$\frac{1}{\varepsilon_\lambda}\,\left.\frac{d^2\varepsilon_\lambda}{dt^2}\right|_{T_{\lambda,\mathrm{pk}}} \;=\; -\,\frac{2\mu^2\omega^2}{\sigma^2}\,\left(\frac{T_{\lambda,\mathrm{pk}}}{T_*}\right)^{2/\mu} \quad . \tag{35}$$

equating with with $2/\Delta\tau_\lambda^2$ yields the emission lifetime

$$\boxed{\Delta\tau_\lambda \;=\; \frac{\sigma}{\omega\mu}\,\left(\frac{T_{\lambda,\mathrm{pk}}}{T_*}\right)^{-1/\mu} \quad ,} \tag{36}$$

This matches eq. (8) for an index

$$\boxed{\nu \;=\; \frac{1}{\mu} \;=\; \frac{3+2\alpha}{4} \quad .} \tag{37}$$

g. The given values correspond to $L_9 = 5$ and $\mathcal{E}_9 = 2000$. Using the latter in eq. (14) yields

$$T_{6,*} = (0.77 \cdot 2000)^{1/3} = 11.5 \quad . \tag{38}$$

From eq. (29) we find

$$\beta = e^{2\sigma^2} \quad , \tag{39}$$

which yields a different value, greater than one, for each spectral line according to its value of σ. These are listed in table 2. Multiplying these by the different values of T_λ and then dividing by $T_* = 11.5$ MK yields the ratio $T_{\lambda,\mathrm{pk}}/T_*$ also listed in the table. The first ratio, for Fe XXI, is greater than one, meaning the 128.7Å line will *not* peak during the radiative phase. According to the wording of the problem, we need not compute a time for this line. (If we tried we would find a *negative* cooling time — strange!).

For the remaining three spectral lines, the peak time is found by setting $T = T_{\lambda,\mathrm{pk}}$ in eq. (24) and solving for time. The result is

$$t_\lambda = \frac{1}{\omega} \left[\left(\frac{T_{\lambda,\mathrm{pk}}}{T_*} \right)^{-1/2} - 1 \right] \quad . \tag{40}$$

Equation (23) yields

$$\frac{1}{\omega} = (224\,\mathrm{s}) \cdot \frac{12}{2} \cdot \frac{5}{(0.77 \cdot 2000)^{1/6}} = 1.9 \times 10^3\,\mathrm{s} \quad . \tag{41}$$

This is the characteristic cooling time for this flare loop. It is reassuringly comparable to cooling times we often encounter: some fraction of an hour (here it is 31 min. 40 s).

Combining this values with eq. (40) yields the delays given in table 2. The lifetime is given by

$$\Delta\tau_\lambda = \frac{\sigma}{\omega\mu} \left(\frac{T_{\lambda,\mathrm{pk}}}{T_*} \right)^{-2/\mu} = (1.0 \times 10^3\,\mathrm{s})\,\sigma \left(\frac{T_{\lambda,\mathrm{pk}}}{T_*} \right)^{-1} \quad . \tag{42}$$

ion	λ Å	β	$T_{\lambda,\mathrm{pk}}/T_*$	t_λ s	$\Delta\tau_\lambda$ s
Fe XXI	128.7	1.20	1.20	–	–
Fe XVIII	93.9	1.47	0.88	134	463
Fe XIV	211.3	1.13	0.20	2,500	560
Fe IX	171.1	1.42	0.10	4,200	1,300

Table 2: Derived parameters for 4 different ions.