HW Set 2 Problem 1

In this problem, we will modify (simplify) the configuration in Problem Set 1, with the magnetic field generated by a bipole of strength λ at the position in the yz plane $(\pm b, 0)$, λ, b both being positive, and a line current I at the position $(0, h)$ and its mirror current $-I$ at $(0, -h)$. This produces a simple 2-dimensional flux rope with an X-point below the flux rope (the center of which is the line current) with an appropriate sign of the current I. With this toy model, we will predict observational measurements, reconnection rate and coronal dimming, as the flux rope rises.

(a) [5 points] As the flux rope rises, we assume that the current I in the flux rope is constant, and reconnection at the X-point below the rope is always very efficient so that a current sheet never forms there. With such a configuration, find the reconnection flux as a function of the flux rope height h. For a high-lying rope $h >> b$, and also given $\lambda = 2I/c$, find the leading term of your solution.

This problem is a simplified version of Problem Set 1, where the quadrupole is replaced by a bipole; therefore, the flux function of the potential field by the bipole is given by:

$$A_p = \lambda \left[\mathrm{atan}\left(\frac{y+b}{z}\right) - \mathrm{atan}\left(\frac{y-b}{z}\right) \right]. \tag{1}$$

The flux function by the line current I at $(0, h)$ and its image current $-I$ at $(0, -h)$ is given by

$$A_I = \frac{I}{c} \ln \left[\frac{(z+h)^2 + y^2}{(z-h)^2 + y^2)} \right]. \tag{2}$$

And the total flux function is $A = A_p + A_I$.

If I flows in the x-direction, an X-point is present below the flux rope where the magnetic field becomes zero. We can find this point by

$$B_y = \frac{\partial A}{\partial z}_{y=0} = -\frac{2\lambda b}{z^2 + b^2} - \frac{2I}{c}\frac{2h}{z^2 - h^2} = 0. \tag{3}$$

Now given that $\lambda = 2I/c$, we can solve the quadratic equation to get the height of the X-point to be

$$z_x = \pm \sqrt{\frac{bh(h-b)}{h+b}}. \tag{4}$$

For the solution above the photosphere in the corona, we take only the positive root. One can see that at the limit $h >> b$, the above solution becomes $z_x \approx \sqrt{bh}$.

The flux function at this point is given by

$$\psi(h) = A(0, z_x(h)) = 2\lambda \mathrm{atan}\left(\frac{b}{z_x}\right) + \frac{I}{c} \ln \left[\frac{(z_x+h)^2}{(z_x-h)^2} \right] = \lambda \left[2\mathrm{atan}\left(\frac{b}{z_x}\right) + \ln \left(\frac{h+z_x}{h-z_x}\right) \right], \tag{5}$$

with $z_x(h)$ given in Eq. 4. At the limit $h >> b$, it can be found that the leading term is

$$\psi(h) = 4\lambda\sqrt{\frac{b}{h}}. \tag{6}$$

As the flux rope rises, the X-point is rising, and the total flux below the X-point increases due to reconnection. Since the flux function on the photosphere surface does not change, the difference between the flux $\psi(h)$ and the flux at the initial height $\psi(h(0))$ is the total reconnectionn flux $\psi_R(h) = \psi(h(0)) - \psi(h)$. In the limit $h >> b$, the reconnection flux is approximated by

$$\psi_R(h) = 4\lambda\left(\sqrt{\frac{b}{h(0)}} - \sqrt{\frac{b}{h}}\right). \tag{7}$$

Note that the reconnection flux is the difference between the flux at a given time and the flux at the initial time. The above format gives the absolute value of the reconnection flux.

(b) [3 points] The measured CME velocity near the Sun often exhibits a time evolution that may be described by a hyperbolic function

$$v = \frac{v_0}{2}\left[\tanh\left(\frac{t-2\tau}{\tau}\right) + 1\right],$$

where t is time, τ is a time constant, and v_0 is the peak velocity the CME attains. Use the following set of parameters, $v_0\tau = b$ and $h(t = 2\tau) = 5b$, for two different values of the time constant $\tau = 50, 150$ sec, respectively, plot the reconnection flux ψ_R versus time together with the velocity, with the time range from 0 to 1000 sec.

We can find the height of the flux rope by a time integral of $v(t)$, which yields

$$h = \int_0^t v dt = \frac{b}{2}\left[\frac{t}{\tau} + \ln\left(\cosh\left(\frac{t-2\tau}{\tau}\right)\right)\right] + 4b. \tag{8}$$

Taking this expression back to Eq. 7, we can plot reconnection flux $\psi_R(h(t))$ as seen in the top panels of the figures, together with the velocity time profile. It is seen that the rise of the reconnection flux is delayed relative to the flux rope velocity.

Now recall the Problem Set 1, the reconnection flux here is really a two-dimensional measurement with the translational lengthscale L_x not specified. If we use the same values of λ and L_x as in Problem Set 1, the reconnection flux in this problem is of order 10^{21} Mx (G cm), well within the range of reconnection flux measured in flare observations, which is of order 10^{20-22} Mx.

(c) [3 points] Find and plot the reconnection rate $\dot{\psi}_R = d\psi_R/dt$ and rope's acceleration. Evaluate the time difference between the peak acceleration and peak reconnection rate for the two values of the time constant τ.

From the hyperbolic function of the flux rope velocity profile, the flux rope acceleration is simply

$$a = \frac{dv}{dt} = \frac{v_0}{2\tau}\left[1 - \tanh^2\left(\frac{t-2\tau}{\tau}\right)\right] = \frac{2v}{\tau}\left(1 - \frac{v}{v_0}\right), \tag{9}$$

which peaks at $t_a = 2\tau$ or $v_a = v_0/2$. On the other hand, the reconnection rate is given by

$$\dot{\psi}_R = \frac{d\psi_R}{dt} = \frac{dh}{dt}\frac{d\psi_R}{dh} = 2\lambda v\sqrt{\frac{b}{h^3}}, \tag{10}$$

which decreases with the height and increases with the velocity. Therefore, the faster the rope rises, the earlier the reconnection rate peaks. Now if we compare the two rates, the term $(1 - v/v_0)$ decreases faster than the term $h^{-3/2}$, therefore, the flux rope acceleration peaks ahead of the reconnection rate, but the time difference becomes smaller as the rope rises faster (smaller τ). Indeed, with a little bit algebra and abiding by the $h >> b$ assumption, it can be found that the reconnection rate peaks at the time when the CME velocity satisfies

$$v_0 > v_\psi = \frac{v_0}{1 + \frac{3}{4}\frac{b}{h}} > \frac{1}{2}v_0. \tag{11}$$

It is clear that, in this specific example, reconnection rate is going to peak **after** the peak acceleration. Working our way further out, we can find the time t_ψ of peak $\dot{\psi}_R$ which satisfies

$$\tanh\left(\frac{t_\psi - 2\tau}{\tau}\right) = \frac{2}{1 + \frac{3}{4}\frac{b}{h}} - 1. \tag{12}$$

With the b/h value, $\Delta t = t_\psi - t_a \sim \tau$. You can find this from your plots of a and $\dot{\psi}_R$ with two different values of τ, as shown in the middle panels of the figures.

Again, with parameters same as in the problem set 1, the reonnection rate in this problem is of order 10^{18} Mx/s, consistent with observationally measured values. We can probe a little bit the CME kinematics as well by assuming that $v_0 = 1000$km/s, a typical peak velocity of a fast CME. With the given parameters, the initial height of the flux rope is about 10^{10} cm, a fraction of the solar radius. The CME acceleration in this problem is quite large, of order 10 km/s, given the short time-scale in this problem.

(d) [3 points] As the rope rises, when we look down upon the rising rope, we may observe coronal dimming. Suppose that the rope is truely 2-dimensional with reconnection injecting bubble field lines into the rope, and the axial flux (which is along the x-direction in this 2d model) is conserved, but is somehow bent and firmly attached to the surface at two locations. Suppose that coronal mass along this axial direction is conserved and the total length of the axial flux tube grows proportionally with h, for a uniform adiabatic expansion, calculate and plot the time evolution of the density (n), temperature (T), and total emission measure (EM) along this axial flux tube compared with their initial values. You may treat the coronal mass as an ideal gas with the ratio of specific heats $\gamma = 5/3$.

From mass conservation, it is easily seen that $n(t)l(t) = n(0)l(0)$, therefore,

$$\frac{n(t)}{n(0)} = \frac{l(0)}{l(t)} = \frac{h(0)}{h(t)}. \tag{13}$$

The total emission measure $EM = n^2 l = n(0)l(0)n(t)$, so there is

$$\frac{EM(t)}{EM(0)} = \frac{n(t)}{n(0)} = \frac{h(0)}{h(t)}. \tag{14}$$

The temperature can be found from the adiabatic relation

$$T(t)l(t)^{\gamma-1} = T(0)l(0)^{\gamma-1}, \tag{15}$$

leading to

$$\frac{T(t)}{T(0)} = \left[\frac{h(0)}{h(t)}\right]^{\gamma-1}. \tag{16}$$

The time evolution of density, emission measure, and temperature is given in the bottom panels of the figures for the time constant $\tau = 50, 150$ sec.

(e) [2 points] The coronal temprature before the eruption is 1.5 MK. Suppose that we observe the feet of the flux rope with the Fe XII line at 193Å, and the temperature response of the instrument for this line is given by a Gaussian

$$G(T) = \exp\left[-\frac{(T-T_0)^2}{2T_w^2}\right],$$

where $T_0 = 2T_w = 1.5$MK. The observed counts is an integral along the line of sight with the effective length scale $l_{eff} \sim h$. Find and plot the time sequence of the observed data counts normalized to the pre-eruption data counts (aka, coronal dimming) at this line. Evaluate the degree of dimming when the flux rope is twice as high as its initial height.

Here we need to consider the temperature as well as the density change of the plasma. For the uniform expansion, the differential emission measure is a Dirac-delta function $n(T')^2 dl/dT' \propto n(T')^2\delta(T'-T)$. Therefore, the observed data counts at a given time is given by

$$I(t) = \int n^2 \frac{dl}{dT'} G(T')dT' = n(T(t))^2 l(T(t))G(T(t)), \tag{17}$$

which is normalized to the pre-eruption emission by

$$\frac{I(t)}{I_0} = \frac{n(t)^2 l(t)G(T(t))}{n(0)^2 l(0)G(T(0))} = \frac{n(t)G(T(t))}{n(0)G(T(0))} = \frac{h(0)G(T(t))}{h(t)G(T(0))}, \tag{18}$$

where we used the mass conservation relation again. With the given parameters and Eq. 16, it is seen that $G(T(0)) = G(T_0) = 1$, and $G(T(t)) = \exp[-2(\bar{h}^{\gamma-1}-1)^2]$ where $\bar{h} \equiv h(0)/h(t)$ by definition. Therefore,

$$\frac{I}{I_0} = \bar{h}\exp[-2(\bar{\mathrm{h}}^{\gamma-1}-1)^2]. \tag{19}$$

The time evolution is plotted in the bottom panels of the figures. When the rope is twice as high, $\bar{h} = 0.5$, the temperature of plasma is $T = \bar{h}^{\gamma-1}T_0 = 0.62T_0$. At this temperature, $G(T) = \exp[-2(0.62-1)^2] = 0.75$. Meanwhile the density or emission measure $n^2 l$ is reduced by one half. Therefore, the observed counts is only 0.38 of the pre-eruption emission. Such an extent of dimming is consistent with many observations.

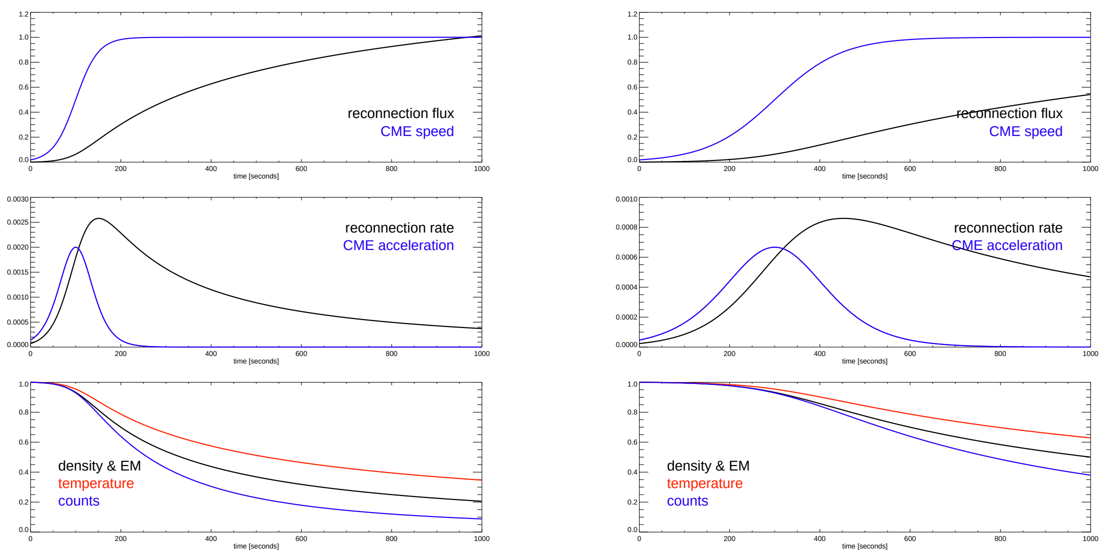

Fig. 1.— Eovlution of the magnetic and plasma properties of the flux rope with $\tau = 50$ sec (left) and $\tau = 150$ sec (right). Top: reconnection flux (in units of λ) and CME velocity (normalized to v_0). Middle: reconnection rate (in units of λ per second) and CME acceleration (normalized to $5v_0$). Bottom: plamsa number density, Emission Measure (EM), and temperature along the axial flux tube, and observed counts at the feet. All properties are normalized to their values at the initial time.