Hale COLLAGE problem set 1 (Due Wed. Feb 15)

1. The first lectures used simple problems, formulated in two dimensions, to illustrate the role of magnetic reconnection in releasing stored energy. Here we explore such model for a compact flare — no flux rope or eruption. The model involves two identical magnetic bipoles, whose field is shown in fig. 1a. Any magnetic field in the (y, z) plane, can be written using a flux function $A(y, z)$

$$\mathbf{B}(y,z) \;=\; \nabla A\times\hat{\mathbf{x}} \;=\; \frac{\partial A}{\partial z}\,\hat{\mathbf{y}} \;-\; \frac{\partial A}{\partial y}\,\hat{\mathbf{z}} \quad . \tag{1}$$

Contours of the function show field lines, and its value at any point ($z \geq 0$) gives the flux (per ignorable length) ψ passing between that point and the origin $(y, z) = (0, 0)$. (If you have no experience working with flux functions, you might try convincing yourself of these important properties using vector calculus.)

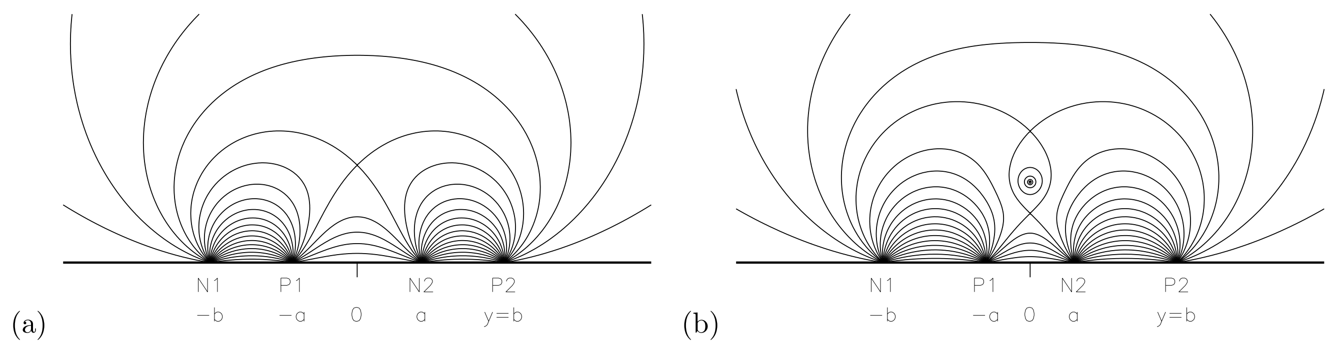

Figure 1: The field above a pair of line-bipoles $P1$–$N1$ (left) and $P2$–$N2$ (right). (a) Shows the potential field defined by eq. (2). (b) Shows the field with a single island, given by eq. (3), used as a simple model of a current sheet. The flux linking $P1$ to $N2$, denoted ψ_{12} is the same in both examples.

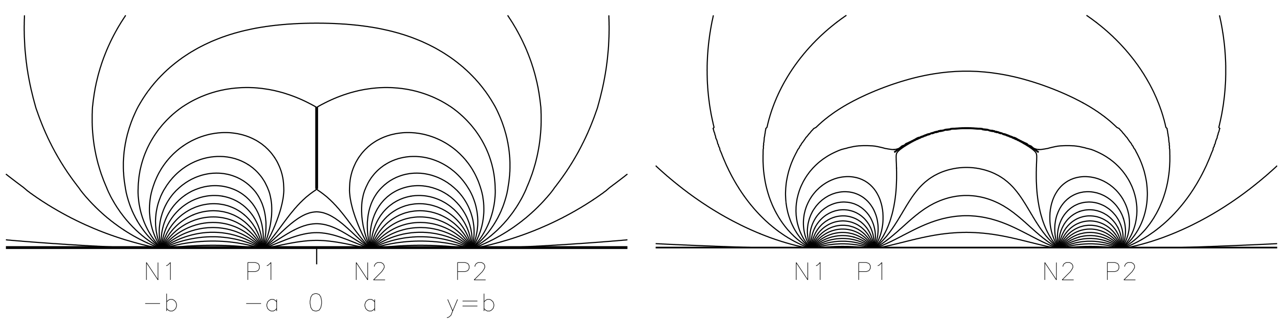

Figure 2: The field when the inner separation, $2a$, is decreased (left) and increased (right). The null point has been deformed into a current sheet, shown with a dark curve. The flux ψ_{12} connected $P1$ to $N2$ is the same in each case as in the potential field, fig. 1a.

The pair of bipoles consists of four sources, two positive and two negative, located on the photosphere ($z = 0$) and arranged symmetrically about the z axis ($y = 0$) at points

$y = \pm a$ and $y = \pm b$ (see fig. 1a). The potential field, $\nabla \times \mathbf{B} = 0$, above this distribution is generated by the flux function

$$
\begin{aligned}
A^{(p)}(y, z) \quad &= \quad \lambda \tan^{-1}\left(\frac{y+b}{z}\right) \quad - \quad \lambda \tan^{-1}\left(\frac{y+a}{z}\right) \\
&\quad + \lambda \tan^{-1}\left(\frac{y-a}{z}\right) \quad - \quad \lambda \tan^{-1}\left(\frac{y-b}{z}\right) \quad , \qquad (2)
\end{aligned}
$$

for $z > 0$, where the magnetic flux (per ignorable length) of each sources in $\pi\lambda$

a. The potential field has a single X-point at $(y, z) = (0, z_x)$. Find the value of z_x explicitly in terms of a, b, and λ. Evaluate the flux function at that point to obtain the amount of flux, $\psi_{12}^{(p)}$, connecting $P1$ to $N2$ in a potential field. Verify that this expression has the correct values in limits $a/b \to 1$ and $a/b \to 0$.

b. Say we begin with $a = b/2$, and then change a by a small amount $\Delta a \ll a$. Compute the change $\Delta\psi_{12}^{(p)}$ to lowest order in $\Delta a/a \ll 1$.

c. In an ideal plasma, where $E' = 0$, the flux connecting $P1$ to $N2$, ψ_{12}, will not change. If we change a, as in part b, the field will develop a current sheet in place of the null point, as shown in fig. 2. It is possible to approximate the sheet with a set of wires whose islands link together to form a chain. The simplest such model has a *single* wire, located at $z = h$, and carrying current $I_{\rm cs}$. This creates a single island, as shown in fig. 1b. The field is given by a potential

$$
A(y, z) \quad = \quad A^{(p)}(y, z) \; + \; \frac{I_{\rm cs}}{c} \, \ln\left[\frac{y^2 + (z+h)^2}{y^2 + (z-h)^2}\right] \quad , \qquad (3)
$$

where $A^{(0)}(y, z)$ is given by eq. (2). (Note that the added term vanishes along $z = 0$, so the wire does not affect the vertical photospheric field. This is because the added terms consists of a wire, at $z = h$, and an oppsing *image current* at $z = -h$.)

The field generated by eq. (3), contains two null points in place of the single null in $A^{(0)}(y, z)$. To model a current sheet these *must* occur at the same value of A, as in fig. 1b. This value gives the flux ψ_{12} in the presence of the current sheet. The separation between the two nulls approximates the extent of the actual current sheet, $2L$, as in fig. 2.

For which sign of $I_{\rm cs}$ will these null points both fall on the z axis as in fig. 1b? Explain your answer in words. Use this sign and find the null positions in the limit of small current. In this limit you may take $h = z_x$, from part a., and expand expressions in powers of $z - z_x$, to obtain a location to leading order in $I_{\rm cs}/c\lambda \ll 1$. Find both nulls and verify that they each have the same value of A (justifying our assignment of $h = z_x$).

d. Use the results of c. to compute the flux difference from a potential field in the case $b = a/2$ to leading order in $I_{\rm cs}$. This should take a form

$$
\Delta\psi_{12} \;=\; \psi_{12} \;-\; \psi_{12}^{(p)} \;\propto\; I_{\rm cs}\,\ln\Big(\alpha|I_{\rm cs}|\Big) \quad , \qquad (4)
$$

where you need to find the constant of proportionality *and* the value of α. Next find the distance between X-points, as an approximation to the full length of the current sheet, $2L$. Find this to leading order in current, explicitly in terms of $I_{\rm cs}$, λ and a.

e. The energy (per ignorable length) released by complete reconnection of a current sheet ($\Delta\psi_{12} \to 0$) can be found from the electromagnetic work integral

$$\Delta\mathcal{E}_M \;=\; \frac{1}{c}\int\limits_0^{\Delta\psi_{12}} I_{\rm cs}(\Delta\psi_{12})\,d(\Delta\psi_{12}) \quad . \tag{5}$$

Use the approximate expression from part d. to perform the integral explicitly and obtain an explicit expression in terms of $I_{\rm cs}$. (This might require an integration by parts.)

f. We may use this simple model to obtain the magnetic field strength B_i just outside the actual current sheet. Ampère's law,

$$\frac{4\pi|I_{\rm cs}|}{c} \;=\; \oint \mathbf{B}\cdot d\mathbf{l} \;\simeq\; 4L\,B_i \quad , \tag{6}$$

can be used, in conjunction with the length and current from e., to obtain an explicit expression for B_i in terms of $I_{\rm cs}$, λ and a, for the case $b = 2a$. Assuming some uniform mass density ρ_0 write down the value of the Alfvén-transit time

$$\tau_{\rm A} \;=\; \frac{2L}{v_A} \quad , \tag{7}$$

for the current sheet.

g. We now use the results above to obtain numerical values for a simplified model of a compact flare. Consider a case where $a = 3\times10^9$ cm, and $b = 2a = 6\times10^9$ cm. We will assign a finite extent, $L_x = 10^{10}$ cm in the previously ignorable direction, but continue to use the two-dimensional expressions obtained above. Assign the parameter λ so that every source has a total flux of 10^{22} Mx. The current sheet builds up as the inner sources each move by $\Delta a = 10^9$ cm under *ideal* conditions ($E' = 0$). Assume the motion is in the direction which produces a *vertical* current sheet. Use the expressions derived for $\Delta a \ll a$ to find the length $2L$ of the resulting current sheet, the height of its center, h, the current it carries $|I_{\rm cs}|$ (express this in Amps), and the energy available for release by magnetic reconnection. Finding the current will require you to solve a transcendental equation. You may do this approximately in whichever manner you prefer.

h. Following the storage phase (part g.) there is sudden and complete reconnection of the current sheet, restoring a potential field ($\Delta\psi_{12} \to 0$). This produces flare ribbons in a region whose photospheric, vertical field strength is $B_{z,0} = 300$ G (use this instead of the pathological values you would obtain using eq. (3) at $z = 0$). Each flare ribbon moves horizontally with a mean velocity $v_{\rm rib} = 3$ km/s. What is the time, $\tau_{\rm rx}$, required for complete reconnection? Use this to compute the average power released in the flare, the mean reconnection electric field (expressed in V/m), and the energy flux incident on each ribbon (erg/cm^2/s). Finally assume a mass density $\rho_0 = 10^{-15}$ g/cm^3 at the current sheet and compute the Alfvén Mach number of the reconnection.

Solution:

a. It is easy to show that the vertical field $B_z = -\partial A^{(p)}/\partial y$ vanishes at $y = 0$. The horizontal field at $y = 0$ is

$$B_y^{(p)}(0,z) \;=\; \left.\frac{\partial A^{(p)}}{\partial z}\right|_{y=0} \;=\; \frac{2\lambda a}{z^2+a^2} \;-\; \frac{2b\lambda}{z^2+b^2} \;=\; 2\lambda\frac{(a-b)(z^2-ab)}{(z^2+a^2)(z^2+b^2)} \quad . \qquad (8)$$

This vanishes at the point

$$z_x \;=\; \sqrt{ab} \quad , \qquad (9)$$

independent of λ. Putting this into eq. (2) yields the flux

$$\psi_{12}^{(p)} \;=\; A^{(p)}(0,z_x) \;=\; 2\lambda\left[\tan^{-1}\left(\sqrt{\frac{b}{a}}\right) \;-\; \tan^{-1}\left(\sqrt{\frac{a}{b}}\right)\right] \quad . \qquad (10)$$

If $a/b \to 1$ this clearly goes to $\psi_{12}^{(0)} \to 0$. In this limit the two bipoles become isolated from one another, and *share no flux.* In the other limit, $a/b \to 0$, we find $\psi_{12}^{(0)} \to \pi\lambda$, which is all the flux in either source. This represents *complete* connection between $P1$ to $N2$. ($P2$ connects to completely $N1$ over the top.)

b. The change in flux will be

$$\frac{\partial\psi_{12}^{(p)}}{\partial a} \;=\; -\,2\sqrt{\frac{b}{a}}\,\frac{\lambda}{a+b} \quad . \qquad (11)$$

So the change, to lowest order, will be

$$\Delta\psi_{12}^{(p)} \;=\; \Delta a\,\frac{\partial\psi_{12}^{(0)}}{\partial a} \;=\; -\,\frac{2\sqrt{ab}\,\lambda}{a+b}\,\frac{\Delta a}{a} \quad . \qquad (12)$$

Evaluating this at $b = 2a$ yields

$$\Delta\psi_{12}^{(p)} \;=\; -\,\frac{2\sqrt{2}\,\lambda}{3}\,\frac{\Delta a}{a} \quad . \qquad (13)$$

c. Along the z axis, the potential magnetic field points rightward ($B_y > 0$) low down and leftward higher up. To have null points along the axis, the wire must produce a field capable of canceling these both. It must be *leftward* below and *rightward* above. We see, from simple application of the right-hand rule, that such current must be directed *into* the page: $I_{\rm cs} < 0$.

With the addition of the wire the horizontal field on the z axis is

$$\begin{aligned} B_y(0,z) \;&=\; \left.\frac{\partial A}{\partial z}\right|_{y=0} \;=\; \frac{2a\lambda}{z^2+a^2} \;-\; \frac{2b\lambda}{z^2+b^2} \;-\; \frac{4I_{\rm cs}}{c}\,\frac{h}{z^2-h^2} \\ &=\; \frac{2\lambda a}{z^2+a^2} \;-\; \frac{2\lambda b}{z^2+b^2} \;-\; \frac{4I_{\rm cs}}{c}\,\frac{h}{z^2-h^2} \quad . \end{aligned} \qquad (14)$$

This expression can be put over a common denominator to obtain a numerator quadratic in z^2. The exact locations of the null points are the two roots of this quadratic.

To apply the approximations described, however, it is more convenient to combine the first two terms as $B_y^{(p)}$, given by eq. (8). We then expand this about $z = z_x$ to obtain

$$B_y(0,z) \;\simeq\; \overbrace{B_y^{(p)}(0,z_x)}^{=0} \;+\; \left.\frac{\partial B_y^{(p)}}{\partial z}\right|_{z_x} (z - z_x) \;+\; \cdots \;-\; \frac{4I_{\rm cs}}{c}\frac{z_x}{2z_x(z-z_x)} \quad , \tag{15}$$

where the first term vanishes at $z = z_x$, because that is the potential-field null point. The coefficient in the second term is

$$\left.\frac{\partial B_y^{(p)}}{\partial z}\right|_{z_x} \;=\; -\frac{4\lambda a z_x}{(z_x^2+a^2)^2} \;+\; \frac{4\lambda b z_x}{(z_x^2+b^2)^2} \;=\; -\frac{4\lambda}{(a+b)^2}\left[\sqrt{\frac{b}{a}} \;-\; \sqrt{\frac{a}{b}}\right] \quad , \tag{16}$$

after using $z_x = \sqrt{ab}$. This expression is negative since $b > a$ (and it is evident that $B_y^{(p)}$ decreases with height, going from positive to negative at $z = z_x$) so we introduce the positive factor

$$B'^{(p)} \;\equiv\; \frac{4\lambda}{(a+b)^2}\left[\sqrt{\frac{b}{a}} \;-\; \sqrt{\frac{a}{b}}\right] \;=\; \frac{4\lambda(b-a)}{(a+b)^2\sqrt{ab}} \quad . \tag{17}$$

In terms of this eq. (15) becomes

$$B_y(0,z) \;=\; -(z-z_x)\,B'^{(p)} \;-\; \frac{2I_{\rm cs}}{c}\frac{1}{z-z_x} \quad , \tag{18}$$

which can be recast as a simple quadratic

$$(z-z_x)^2 \;=\; -\frac{2I_{\rm cs}}{c\,B'^{(p)}} \quad . \tag{19}$$

This has two roots, $z_{1,2}$, given by

$$z_{1,2} \;=\; z_x \;\pm\; \sqrt{-\frac{2I_{\rm cs}}{c\,B'^{(p)}}} \;=\; z_x \;\pm\; \frac{(ab)^{1/4}(a+b)}{\sqrt{b-a}}\sqrt{-\frac{I_{\rm cs}}{2c\,\lambda}} \quad , \tag{20}$$

after using eq. (17). Clearly this expression requires $I_{\rm cs} \leq 0$, which we had decided above would be necessary for the null points to lie on the z axis. We note that the lower null point will be in the corona only when

$$\frac{|I_{\rm cs}|}{c} \;<\; \tfrac{1}{2}z_x^2 B'^{(p)} \;=\; \tfrac{1}{2}ab\,B'^{(p)} \quad . \tag{21}$$

This limit will be, however, well outside our assumption of very small current and nulls very close to z_x.

Our assignment $h = z_x$ was premised on the need to make $A(0,z)$ the same at both null points. We verify this by direct substitution and expansion

$$A(0,z_{1,2}) \;=\; A^{(p)}(0,z_x) \;+\; \overbrace{\left.\frac{\partial A^{(p)}}{\partial z}\right|_{z_x}}^{B_y^{(p)}=0}(z_{1,2}-z_x) \;+\; \tfrac{1}{2}\left.\frac{\partial B_y^{(p)}}{\partial z}\right|_{z_x}(z_{1,2}-z_x)^2 \;+\; \cdots$$

$$+\; \frac{I_{\rm cs}}{c}\,\ln\left[\frac{(2z_x)^2}{(z_{1,2}-z_x)^2}\right] \quad . \tag{22}$$

Since the term linear in $(z_{1,2} - z_x)$ vanishes, as indicated, the expression depends only on $(z_{1,2} - z_x)^2$, which is the same for both — see eq. (19). Thus the flux function has the same value at both null points, as required.

d. We can use eq. (20) to compute the length of the current sheet

$$2L \;=\; z_1 - z_2 \;=\; 2\sqrt{\frac{2|I_{\rm cs}|}{c\,B'^{(p)}}} \quad . \tag{23}$$

The value of the flux at either X-point is found from expression (22),

$$\psi_{12} \;=\; \psi_{12}^{(p)} \;+\; \frac{I_{\rm cs}}{c} \;-\; \frac{I_{\rm cs}}{c}\ln\left[\frac{|I_{\rm cs}|}{2abc\,B'^{(p)}}\right] \;=\; \psi_{12}^{(p)} \;-\; \frac{I_{\rm cs}}{c}\ln\left[\frac{|I_{\rm cs}|}{2abce\,B'^{(p)}}\right] \quad , \tag{24}$$

where $e = 2.718$, the base of the natural logarithm, has been taken inside the logarithm. This shows the flux difference

$$\Delta\psi_{12} \;=\; \psi_{12} \;-\; \psi_{12}^{(p)} \;=\; -\,\frac{I_{\rm cs}}{c}\ln\left[\frac{|I_{\rm cs}|}{2abce\,B'^{(p)}}\right] \quad , \tag{25}$$

takes the anticipated form. Since $|I_{\rm cs}|$ is small, the logarithm is negative. Since $I_{\rm cs}$ is also negative the full expression, $\Delta\psi_{12}$, is itself negative. This is the expected contribution from a negative current: it creates flux beneath itself passing from right to left. This continues to be true as long as the current lies within the range given by (21).

Expression (25) does go to zero as $|I_{\rm cs}| \to 0$. Its derivative

$$\frac{\partial(\Delta\psi_{12})}{\partial I_{\rm cs}} \;=\; -\,\frac{1}{c}\ln\left[\frac{|I_{\rm cs}|}{2abc\,B'^{(p)}}\right] \;=\; -\,\frac{1}{c}\ln\left(\frac{L^2}{4ab}\right) \quad , \tag{26}$$

diverges at that point, so the approach to $|I_{\rm cs}| \to 0$ is not simple. The final expression makes use of eq. (23) to show how this singular limit arises from a CS (island) of vanishing diameter. In traditional E&M the derivative of flux w.r.t. current yields the *self-inductance* of the current path — and it is usually a constant. Since $\Delta\psi_{12}$ is a flux-per-unit-length, the derivative (26) will yield a self-inductance-per-unit-length. The final expression is strongly reminiscent of the self-inductance of a loop, of radius $2\sqrt{ab}$, made from wire of diameter $2L$: it depends logarithmically on the ratio of the wire diameter to the loop radius. A wire of vanishing diameter has an infinite self-inductance, since it creates an infinite amount of self flux — mostly in the field diverging at the wire itself. In this case the diameter of the wire is the length of the currenth sheet, $2L$, which itself depends on current. The self-inductance is therefore current dependent, and the current-flux relation, i.e. eq. (25) is non-linear.

Substituting $b = 2a$ into eq. (17) yields

$$B'^{(p)} \;\equiv\; \frac{4\lambda}{9a^2}\sqrt{2}\left[1 \;-\; \tfrac{1}{2}\right] \;=\; \frac{2\sqrt{2}\lambda}{9a^2} \quad . \tag{27}$$

The flux and and current sheet length are then given by

$$\Delta\psi_{12} \;=\; -\,\frac{I_{\rm cs}}{c}\ln\left[\frac{9|I_{\rm cs}|}{8\sqrt{2}\,e\,c\lambda}\right] \quad , \tag{28}$$

$$2L \;=\; a\,3\cdot 2^{3/4}\sqrt{\frac{|I_{\rm cs}|}{c\lambda}} \quad , \tag{29}$$

e. The energy is found by the integral

$$\begin{aligned}\Delta\mathcal{E}_M &= \frac{1}{c}\int\limits_0^{\Delta\psi_{12}} I_{\rm cs}(\Delta\psi_{12})\,d(\Delta\psi_{12}) \;=\; \frac{I_{\rm cs}}{c}\,\Delta\psi_{12} - \frac{1}{c}\int\limits_0^{I_{\rm cs}}\Delta\psi_{12}(I_{\rm cs})\,dI_{\rm cs} \\ &= -\frac{I_{\rm cs}^2}{c^2}\ln\left[\frac{9|I_{\rm cs}|}{8\sqrt{2}\,e\,c\lambda}\right] \;+\; \int\limits_0^{I_{\rm cs}}\frac{I_{\rm cs}}{c^2}\ln\left[\frac{9|I_{\rm cs}|}{8\sqrt{2}\,e\,c\lambda}\right]dI_{\rm cs} \quad . \end{aligned} \tag{30}$$

Provided the current is small, the logarithm will be negative and this energy is positive: reconnection always *releases* energy. This gives

$$\Delta\mathcal{E}_M \;=\; -\frac{I_{\rm cs}^2}{2c^2}\ln\left[\frac{9|I_{\rm cs}|}{8e^{1/2}\sqrt{2}\,c\lambda}\right] \quad . \tag{31}$$

The released energy, $\Delta\mathcal{E}_M$, is not necessarily the same as the energy injected into the corona, via Poynting flux, as the photospheric sources were moved by Δa. That energy can be, and frequently is, negative: the motion *lowers* the energy of the coronal magnetic field. In such cases the energy of the potential magnetic field will decrease by even more than the integrated Poynting flux. Reconnection will then release the energy difference as it allows the potential field to be achieved.

f. According to Ampère's law, the field strength just outside the current sheet is

$$B_i \;=\; \frac{\pi|I_{\rm cs}|}{c\,L} \;=\; \frac{\pi|I_{\rm cs}|}{c}\cdot\frac{2^{1/4}}{3a}\sqrt{\frac{c\lambda}{|I_{\rm cs}|}} \;=\; \frac{2^{1/4}\pi}{3a}\sqrt{\frac{|I_{\rm cs}|\lambda}{c}} \quad , \tag{32}$$

after suing eq. (29) to replace L. The Alfvén time

$$\tau_{\rm A} \;=\; \frac{2L}{B_i/\sqrt{4\pi\rho_0}} \;=\; \frac{(2L)^2\sqrt{\rho_0}}{\sqrt{\pi}|I_{\rm cs}|/c} \;=\; \frac{9\sqrt{8}}{\sqrt{\pi}}\,\frac{a^2\sqrt{\rho_0}}{\lambda} \quad , \tag{33}$$

is, remarkably, independent of the current or the sheet's length. If that seems puzzling, it should. As the current sheet gets longer, it also gets stronger. B_i and L both increase $\propto\sqrt{|I_{\rm cs}|}$, so the Alfvén time is unchanged.

g. A potential stumbling block here arises from trying to apply the two-dimensional model, parts a–f, to a three-dimensional reality. Many quantities in the 2d model are specified on a *per unit length* basis — that is to say per length in the ignorable dimension. So $\psi = \pi\lambda$ is not actually a flux — units of Mx — it is a flux *per length*, with units of Mx/cm = G · cm. Since a real example can extend only a finite distance in this dimension we must divide *actual* quantities by this length. Here it was specified to be $L_x = 10^{10}$ cm. The total flux of a source was given as $\Phi_{N2} = 10^{22}$ Mx. To obtain the two-dimensional linear flux density, $\pi\lambda$, we must divide by L_x to obtain

$$\pi\lambda = \frac{\Phi_{N2}}{L_x} = 10^{12}\,\mathrm{G\,cm} \quad \Longrightarrow \quad \lambda = \frac{\Phi_{N2}}{\pi L_x} \simeq 3\times 10^{11}\,\mathrm{G\,cm} \quad . \tag{34}$$

We use this in eq. (13) to obtain the change in *potential flux*

$$\Delta\psi_{12}^{(p)} = -\frac{2\sqrt{2}\,\lambda}{3}\,\frac{\Delta a}{a} \simeq \frac{2\sqrt{2}}{9}\,\lambda \simeq 10^{11}\,\mathrm{G\,cm} \quad , \tag{35}$$

where we have taken Δa to be *negative* in order to produce a vertical current sheet. In order that the flux ψ_{12} be held at its original value we require that $\Delta\psi_{12} = -\Delta\psi_{12}^{(p)}$. Using eq. (28) we obtain

$$\frac{\Delta\psi_{12}}{\lambda} = -\frac{I_{\mathrm{cs}}}{c\lambda}\ln\left[\frac{9|I_{\mathrm{cs}}|}{8\sqrt{2}\,e\,c\lambda}\right] = -\frac{\Delta\psi_{12}^{(p)}}{\lambda} \simeq -\frac{2\sqrt{2}}{9} \quad . \tag{36}$$

Multiplying the second and fourth expressions by $9/8e\sqrt{2}$ yields the transcendental equation

$$\frac{9I_{\mathrm{cs}}}{8\sqrt{2}\,e\,c\lambda}\ln\left[\frac{9|I_{\mathrm{cs}}|}{8\sqrt{2}\,e\,c\lambda}\right] = \xi\,\ln(|\xi|) = \frac{1}{4e} \simeq 0.1 \quad . \tag{37}$$

It can be easily verified that the solution to this is $\xi = -0.03$, from which we obtain

$$\frac{I_{\mathrm{cs}}}{c} = \frac{8e\sqrt{2}}{9}\,\xi\,\lambda = -0.1\,\lambda \simeq -3\times 10^{10}\,\mathrm{G\,cm} = -3\times 10^{11}\,\mathrm{A} \quad . \tag{38}$$

The value in cgs is

$$I_{\mathrm{cs}} = -0.1\,c\,\lambda = -9\times 10^{20}\,\mathrm{Mx\,s^{-1}} = -9\times 10^{20}\,\mathrm{statamps} \quad . \tag{39}$$

Using the fact that one Amp is 3×10^9 stat amps returns the result from (38).

From eq. (38) we readily obtain the length of the current sheet

$$2L = a\,3\cdot 2^{3/4}\sqrt{\frac{|I_{\mathrm{cs}}|}{c\lambda}} = 1.6\,a = 4.8\times 10^9\,\mathrm{cm} \quad . \tag{40}$$

The center of the sheet is at

$$h = \sqrt{ab} = 4.2\times 10^9\,\mathrm{cm} \quad , \tag{41}$$

so the reconnected loops come to rest at bottom of the sheet, at

$$z_{\mathrm{bot}} = h - L = 1.8\times 10^9\,\mathrm{cm} \quad . \tag{42}$$

Finally, the energy stored in the sheet is

$$\Delta E_M = L_x\,\Delta\mathcal{E}_M = -\frac{I_{\mathrm{cs}}^2\,L_x}{2c^2}\,\ln\!\left(e^{1/2}\xi\right) = \frac{3}{2}L_x\left(\frac{I_{\mathrm{cs}}}{c}\right)^2 = 1.5\times 10^{31}\,\mathrm{erg} \quad . \tag{43}$$

h. The area swept out by one ribbon will be

$$A_{\mathrm{rib}} = \frac{\Delta\psi_{12}L_x}{B_{z,0}} = 3\times 10^{18}\,\mathrm{cm}^2 \quad . \tag{44}$$

The width of the ribbon

$$w_{\rm rib} \;=\; \frac{A_{\rm rib}}{L_x} \;=\; 3\times 10^8\,{\rm cm} \quad , \tag{45}$$

is traversed at a speed of $v_{\rm rib} = 3 \times 10^5$ cm/s, which will require a time

$$\boxed{\tau_{\rm rx} \;=\; \frac{w_{\rm rib}}{v_{\rm rib}} \;=\; 10^3\,{\rm s} \;=\; 17\,{\rm min} \quad .} \tag{46}$$

This creates an electric field

$$\boxed{c\,E_{\rm rx} \;=\; -\frac{d\psi_{12}}{dt} \;=\; -\frac{\Delta\psi_{12}}{\tau_{\rm rx}} \;=\; 10^8\,{\rm G\,cm\,s^{-1}} \;=\; 100\,{\rm V/m} \quad .} \tag{47}$$

The same value is obtained from the ribbon motion

$$c\,E_{\rm rib} \;=\; v_{\rm rib}\,B_{z,0} \;\simeq\; (3\times 10^5\,{\rm cm\,s^{-1}})\,(3\times 10^2\,{\rm G}) \;=\; 10^8\,{\rm G\,cm\,s^{-1}} \quad . \tag{48}$$

This field appears rather manageable — the Earth's atmosphere maintains a field around this strong all the time. It is, however, ridiculously large for a full ionized, high temperature plasma.

The mean reconnection power is

$$\boxed{P_{\rm rx} \;=\; \frac{\Delta E_M}{\tau_{\rm rx}} \;=\; 1.5\times 10^{28}\,{\rm erg\,s^{-1}} \quad .} \tag{49}$$

The initial rate of electrodynamic work is

$$E_{\rm rx}\,|I_{\rm cs}|\,L_x \;=\; c\,E_{\rm rx}\,\frac{|I_{\rm cs}|}{c}\,L_x \;=\; 3\times 10^{28}\,{\rm erg\,s^{-1}} \quad . \tag{50}$$

The current drops to zero, yielding a *mean* power equal to the value from (49). This energy reaches four different ribbons, each with area $A_{\rm rib}$, where it produced a flux

$$\boxed{F_{\rm fl} \;=\; \frac{P_{\rm rx}}{4A_{\rm rib}} \;=\; 1.3\times 10^9\,{\rm erg\,s^{-1}} \quad .} \tag{51}$$

This is a relatively small value, and would produce a mild flare. It is,however, the mean value, and if the reconnection were not steady or uniform, there could be times and places with larger energy flux.

Finally, we use eq. (6) to find the field strength at the current sheet

$$B_i \;=\; \frac{\pi}{L}\,\frac{|I_{\rm cs}|}{c} \;=\; \frac{\pi}{2.4\times 10^9\,{\rm cm}}\,3\times 10^{10}\,{\rm G\,cm} \;=\; 40\,G \quad . \tag{52}$$

This is about one-tenth of the photospheric value because the current sheet is very high in the corona. This gives an Alfvén speed

$$v_{\rm A} \;=\; \frac{B_i}{\sqrt{4\pi\rho_0}} \;=\; 3.6\times 10^8\,{\rm cm/s} \quad , \tag{53}$$

for which the Alfvén transit time is

$$\tau_{\rm A} \;=\; \frac{2L}{v_{\rm A}} \;=\; \frac{4.8\times 10^9\,{\rm cm}}{3.6\times 10^8\,{\rm cm/s}} \;=\; 13\,{\rm sec} \quad . \tag{54}$$

The Alfvén Mach number for the reconnection is therefore

$$M_{\rm Ai} \;=\; \frac{\tau_{\rm A}}{\tau_{\rm rx}} \;=\; 0.013 \quad . \tag{55}$$

This is not an uncommon value for fast reconnection observed at the Sun. The inflow speed will be

$$u_i \;=\; M_{\rm Ai}\, v_{\rm A} \;=\; 4.7\times 10^6\,{\rm cm/s} \quad . \tag{56}$$

The inductive electric field at the edge of the current sheet is therefore

$$c\,E \;=\; u_i\, B_i \;=\; 1.9\times 10^8\,{\rm G\,cm\,s^{-1}} \quad . \tag{57}$$

This is roughly the same as the values obtained by other means. It is notable that while the photospheric field is ten times greater than B_i, the ribbon's velocity is ten times *smaller* than u_i.