Hale COLLAGE problem set 4 (Due Fri. Apr. 14)

We represent the non-thermal component of the electron distribution by the angle-averaged distribution function $f(v)$ defined so that the number density and energy density of non-thermal electrons are

$$n_{nt} = \int_0^\infty f(v)\,dv \quad , \quad \varepsilon_{nt} = \tfrac{1}{2}m_e \int_0^\infty v^2 f(v)\,dv = \tfrac{3}{2}m_e\,n_{nt}\,v_{nt}^2 \quad , \tag{1}$$

where v_{nt} is a characteristic velocity for the non-thermal electrons. The non-thermal distribution function evolves according to the Fokker-Planck equation like that on slide 32 of lecture 18. We will write this as

$$\frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial t}\right)_c = \frac{\partial}{\partial v}\left[\frac{K(v^2-2v_{th}^2)}{v^4}\,f + \left(\frac{Kv_{th}^2}{v^3}+D^{(\mathrm{turb})}\right)\frac{\partial f}{\partial v}\right] \quad , \tag{2}$$

where $K = 4\pi e^4 n_{th}\Lambda/m_e^2$ is a constant, and Λ is the Coulomb logarithm. The density of non-thermal electrons $n_{nt} \ll n_{th}$, the number density of thermal electrons with which the non-thermal electrons collide. It is only this larger density which appears in the collision operator through the constant K. The thermal velocity of the target electrons, $v_{th}^2 = k_bT/m_e$, is also part of the collision operator.

a. Show that when there is no turbulent contribution, i.e. $D^{(\mathrm{turb})} = 0$, a Maxwellian with $v_{nt} = v_{th}$ is a steady solution. Be aware that this is a Maxwellian version of $f(v)$, not $f(\mathbf{v})$ — the difference is explained on slide 29 of lecture 17.

b. Now include an arbitrary turbulent diffusion $D^{(\mathrm{turb})}(v) \neq 0$ and find an expression for the change of non-thermal energy density $\partial\varepsilon_{\mathrm{nt}}/\partial t$. Integrate by parts to obtain an expression involving only moments of f and possibly surface terms. When doing so assume that

$$f(v) \to C\,v^2 \quad , \quad v \to 0 \quad , \tag{3}$$

for some constant C, and that $f(v)$ vanishes more rapidly than any inverse power of v as $v \to \infty$. Assume also that $v^2 D^{(\mathrm{turb})} \to 0$ as $v \to 0$. Surface terms in your final expression will include C. The full expression should contain contributions from collisions, proportional to K, and from turbulence. The latter term should vanish if $D^{(\mathrm{turb})}(v) \propto v^{-1}$.

c. Take $f(v)$ to be a Maxwellian with thermal speed v_{nt}, and show that the collisional contribution to the energy change, found in part b., is proportional to $v_{th}^2 - v_{nt}^2$. If you begin with $v_{nt} > v_{th}$ will collisions increase or decrease the non-thermal energy ε_{nt}?. Explain in words how *elastic* collisions can result in *any* change to energy ε_{nt}.

d. Now take the turbulent diffusion to be of the form which will not change the energy of the non-thermal particles (so the turbulence itself will neither gain nor lose energy to the particles)

$$D^{(\mathrm{turb})} = \frac{G}{v} \quad , \tag{4}$$

where G is a constant. (We showed in lecture 19 that this constant is related to the energy density of the turbulence as $G = (2\pi e/m_e)^2\varepsilon_{\mathrm{turb}}/\bar{k}$, where $\bar{k}$ is a wavenumber characteristic of the turbulent spectrum.) Show that the Fokker-Planck equation is exactly solved by a steady state distribution of the form

$$f(v) \;=\; C\,v^2\left(1+\beta v^2\right)^{-(\delta+1)} \quad , \tag{5}$$

where C, β and δ are all constants. This combines a Maxwellian-like core (i.e. $\beta v^2 < 1$) and power-law tail into a single seamless function The exponent is named δ so that the non-thermal energy flux $F(E) \sim f(v) \to E^{-\delta}$ matching the traditional usage. Write down the values of the constants δ and β, which make expression (5) a steady solution, explicitly in terms of K, G and v_{th}.

e. For the solution found in part d. show that it assumes the form of a Maxwellian in the limit $G \to 0$. Show only that the functional form is correct in that limit, disregarding the normalization constant C. But show also that the Maxwellian obtained in this limit has a width v_{th}.

f. For the solution found in part d. perform integrals in eq. (1) to obtain expressions for C, in terms of n_{nt}, v_{th} and δ. Then find the non-thermal velocity v_{nt}. Show that v_{nt} approaches the expected limit when $\delta \to \infty$. Which values of δ yield a finite value of v_{nt}? The integrals may be performed using

$$I_p \;=\; \int_0^\infty v^{2p}\,(1+\beta v^2)^{-(\delta+1)}\,dv \;=\; \frac{\Gamma\left(\delta-p+\frac{1}{2}\right)\Gamma\left(p+\frac{1}{2}\right)}{2\beta^{(p+1/2)}\,\Gamma(\delta+1)} \quad , \tag{6}$$

where $\Gamma(x)$ is the Γ function defined so that $\Gamma(x+1) = x\Gamma(x)$.

g. Since eq. (5) is a steady equilibrium, the effects of turbulence and collision must cancel at each point in the distribution. The change due to turbulence alone

$$\left(\frac{\partial f}{\partial t}\right)_{\rm turb} \;=\; \frac{\partial}{\partial v}\left(\frac{G}{v}\frac{\partial f}{\partial v}\right) \quad , \tag{7}$$

will either add or subtract particles at a given velocity — collisions will do the opposite. When the distribution has achieved its steady state, over what range of velocities does the turbulence *add* particles? (Naturally it will *remove* the same number from outside this region, since wave-particle interactions will neither create nor destroy particles.)