Hale COLLAGE problem set 3 (Due Fri. Mar. 24)

Flare models include some version of the radiative loss function $\Lambda(T)$. A monomial approximation

$$\Lambda(T) \simeq \Lambda_0\, T_6^{\alpha} \quad , \quad \Lambda_0 = 1.2\times10^{-22}\,\mathrm{erg\,cm^3\,s^{-1}} \quad , \tag{1}$$

permits an analytic treatment. Here T_6 is the temperature expressed in units of Megakelvins, and α is some power-law index. Lecture 9 uses this form with $\alpha = -1/2$, which we argue to be a reasonable approximation to the actual function. Zero-dimensional models must use a version of this function, but it represents the average loss as a function of the average temperature. Once again a monomial approximation permits analytic treatment, but it is not clear that $\alpha = -1/2$ is still appropriate: the average of a function is not the same as the function of the average. We therefore re-derive some of our initial results, but using a general choice of α.

a. Using the form (1) for the radiative loss function, find an expression for the radiative cooling time, τ_{rad}. Express this in terms of T_6 and n_{10}, the electron density in units of $10^{10}\,\mathrm{cm^{-3}}$.

b. Use these results of part a. to find the relation between T_6 and n_{10} for which the radiative and conductive times are equal: $\tau_{\mathrm{rad}} = \tau_{\mathrm{cond}}$. Write n_{10} explicitly in terms of T_6 and L_9, the full length of the loop in units of 10^9 cm. This relations also approximates the condition for mechanical equilibrium. For which α are T and n_e positively correlated with one another, rather than anti-correlated?

c. Under the assumption of Antiochos & Sturrock (1978), evaporation decreases T and increases n_e, keeping constant their product $n_e T$, until $\tau_{\mathrm{rad}} = \tau_{\mathrm{cond}}$, at which point evaporation ends. The constant along which evaporation evolves is set by the total energy (per unit area) released in the flare, $\mathcal{E}$ ($\mathrm{erg\,cm^{-2}}$)

$$n_e\,T \;=\; \frac{\mathcal{E}}{3L\,k_b} \quad . \tag{2}$$

Use this to find the peak density, n_*, achieved at the end of evaporation, along with the corresponding temperature, T_*. Write down expressions for $n_{10,*}$ and $T_{6,*}$, involving only $\mathcal{E}_9$, L_9 and α.

d. Next assume that the radiative cooling process occurs in quasi-static equilibrium, so $\tau_{\mathrm{rad}} = \tau_{\mathrm{cond}}$, as found in part b. Klimchuk *et al* (2008) set the enthalpy equal to the difference between the conductive flux and the radiative loss form the transition region. They further set that loss to be 4 times the coronal radiative loss to obtain an energy equation[1]

$$\frac{3}{2}\frac{dp}{dt} \;=\; -\,5\,n_e^2\,\Lambda(T) \quad , \tag{3}$$

where $p = 2n_e k_b T$. Solve this to find an expression for $T(t)$, resembling slide 29 of lecture 9, but involving an arbitrary index α rather than the choice $\alpha = -1/2$. The solution should take the form

$$T(t) \;=\; T_*\Big(1+\omega\,t\Big)^{-\mu} \quad , \tag{4}$$

[1]The result of the assumptions is that conductive and enthalpy fluxes move energy around without changing it. All changes come from radiative losses which amount to $4+1=5$ times the coronal loss alone.

ion	λ Å	$G_\lambda^{(0)}$ $10^{-25}\,\mathrm{erg\,cm^3\,s^{-1}\,sr^{-1}}$	T_λ MK	σ_λ –
Fe XXI	128.7	1.64	11.51	0.30
Fe XVIII	93.9	1.43	6.91	0.44
Fe XIV	211.3	6.13	1.99	0.25
Fe IX	171.1	37.84	0.82	0.42

Table 1: Parameters defining the function $G_\lambda(T)$ for spectral lines in 4 different ions.

for constants of μ and ω you can write as explicit functions of $\mathcal{E}_9$, L_9 and α. Here we have taken $t = 0$ to be the begining of radiative cooling — the time of peak density. Are there any values of α for which the solution vanishes in finite time?

e. Contribution functions from most optically thin spectral lines appear parabolic on log-log plots (see slides from lecture 11). They may therefore be approximated analytically as

$$G_\lambda(T) \;=\; G_\lambda^{(0)}\,\exp\left[-\frac{\ln^2(T/T_\lambda)}{\sigma_\lambda^2}\right] \quad , \tag{5}$$

where the constants $G_\lambda^{(0)}$, T_λ and σ_λ are constants for that particular spectral line. Values for 4 different spectral lines are given in table 1. Adopting this parameterization, show that during the radiative cooling phase the emissivity of a line, $\varepsilon_\lambda(t)$, will peak at a temperature

$$T_{\lambda,\mathrm{pk}} \;=\; \beta\, T_\lambda \quad , \tag{6}$$

for a factor β dependent only on α and σ_λ. Is β greater than or less than unity? Why? [The easiest approach to this result is to express $\ln(\varepsilon_\lambda)$ as a polynomial in $\ln(T)$, and find the maximum of *that*.]

f. As the loop cools it passes through peak emission at some time t_λ. The lifetime of its emission in this spectral line, $\Delta\tau_\lambda$, can be analytically defined though the relation

$$\left.\frac{1}{\varepsilon_\lambda}\,\frac{d^2\varepsilon_\lambda}{dt^2}\right|_{t_\lambda} \;=\; -\,\frac{2}{\Delta\tau_\lambda^2} \quad . \tag{7}$$

Use the results above to show that

$$\Delta\tau_\lambda \;=\; \frac{\sigma_\lambda}{\omega\mu}\left(\frac{T_{\lambda,\mathrm{pk}}}{T_*}\right)^{-\nu} \quad , \tag{8}$$

for some index ν, depending on α, and ω and μ defined through eq. (4).

g. Now consider a particular flare with $L = 5\times10^9$ cm and $\mathcal{E} = 2\times10^{12}\,\mathrm{erg\,cm^{-2}}$ (the same values used to produce slide 34 of lecture 9). Return to the conventional choice of radiative loss function by setting $\alpha = -1/2$ (consistent with EBTEL). Consider each of the spectral lines in table 1 for which the loop *cools through the peak during its radiative phase.* For each of them find the time and duration of its peak emission, t_λ and $\Delta\tau_\lambda$.