Non-thermal particles

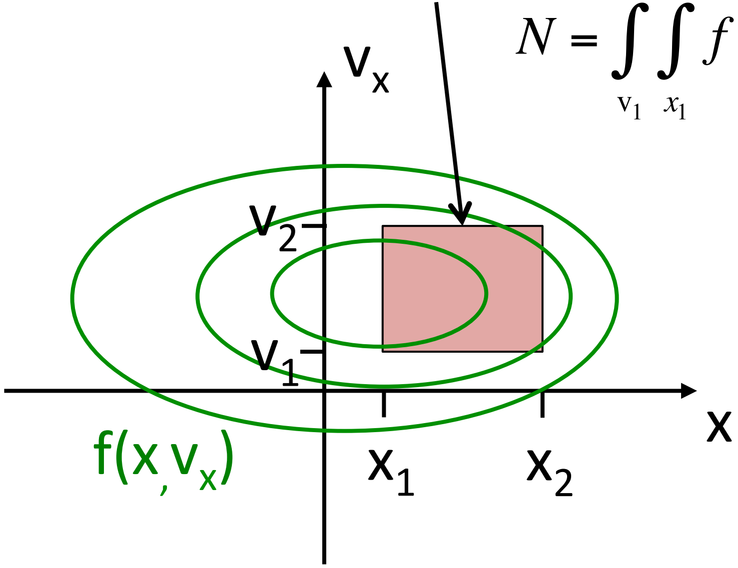
The Fokker-Planck equation

Lecture 18

March 29, 2017

particles in phase-space volume:

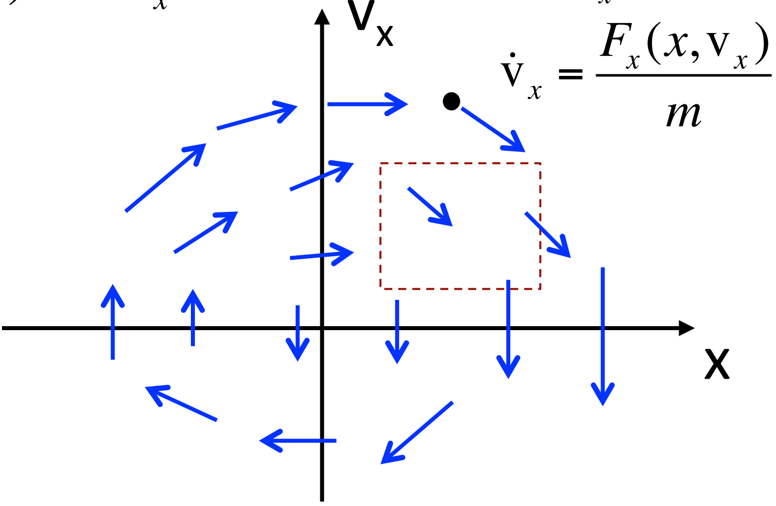
$$N = \int_{v_1}^{v_2}\int_{x_1}^{x_2} f(x, v_x)\,dx\,dv_x$$



particles "flow" through phase-space

$$\dot{x} = v_x$$

$$\dot{v}_x = \frac{F_x(x, v_x)}{m}$$



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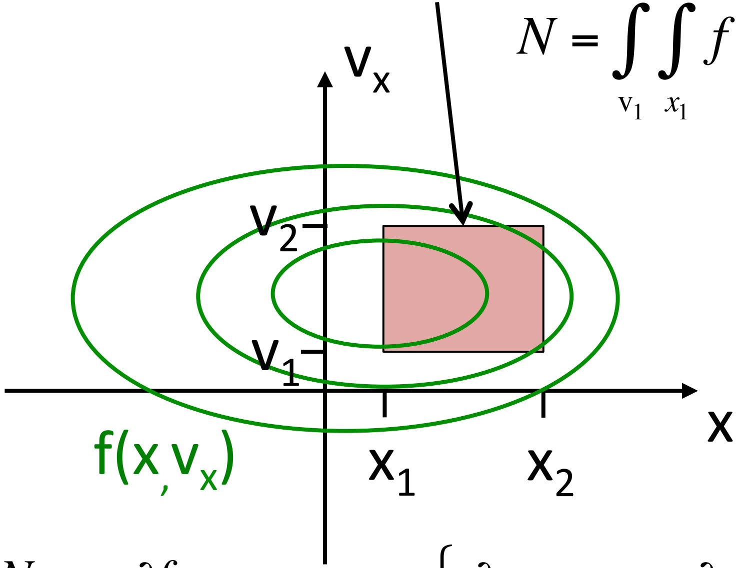

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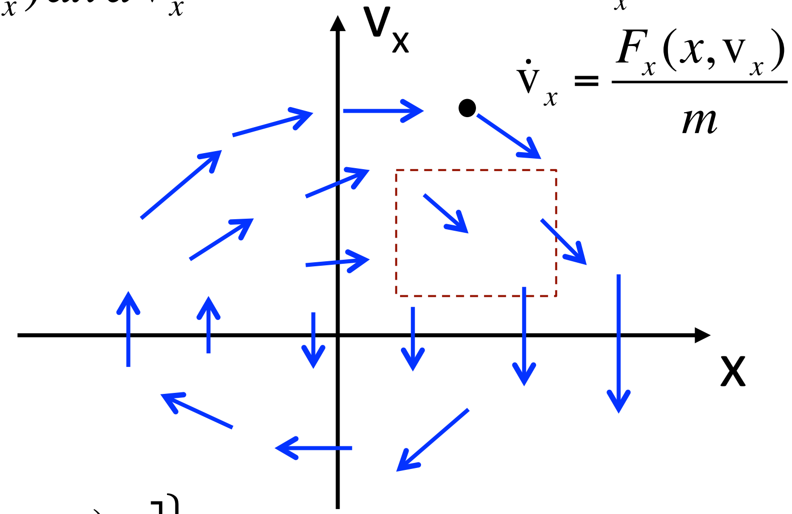
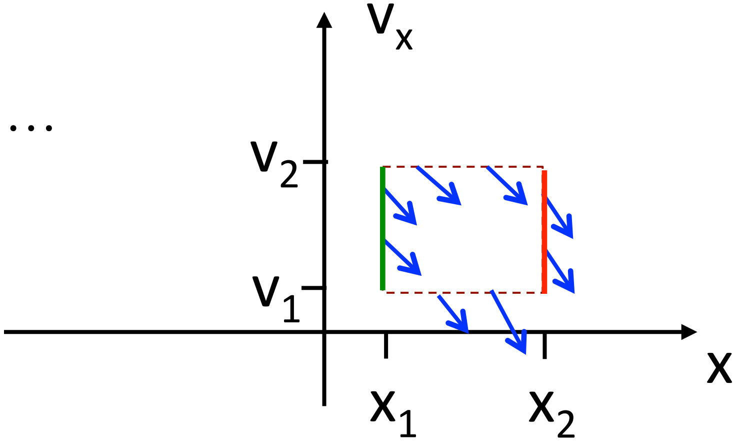


$$\frac{dN}{dt} = \int \frac{\partial f}{\partial t}\,dx\,dv_x = -\int \left\{ \underbrace{\frac{\partial}{\partial x}\left(v_x f\right)} + \frac{\partial}{\partial v_x}\left[\frac{F_x(x,v_x)}{m} f\right]\right\} dx\,dv_x$$

$$= -\int_{v_1}^{v_2}\left[\underbrace{v_x f(x_2,v_x)}_{\text{out}} - \underbrace{v_x f(x_1,v_x)}_{\text{in}}\right] dv_x + \cdots$$

$$\boxed{\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}\left(v_x f\right) - \frac{\partial}{\partial v_x}\left[\frac{F_x(x,v_x)}{m} f\right]}$$



Vlasov's equation

1d version
$$\boxed{\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}\left(\mathrm{v}_x f\right) - \frac{\partial}{\partial \mathrm{v}_x}\left[\frac{F_x(x,\mathrm{v}_x)}{m} f\right]}$$

3d version
$$\frac{\partial f_\sigma}{\partial t} = -\sum_i \frac{\partial}{\partial x_i}\left(\mathrm{v}_i f_\sigma\right) - \sum_i \frac{\partial}{\partial \mathrm{v}_i}\left[\frac{F_i(\mathbf{x},\mathbf{v})}{m} f_\sigma\right] = -\frac{\partial}{\partial x_i}\left(\mathrm{v}_i f_\sigma\right) - \frac{\partial}{\partial \mathrm{v}_i}\left[\frac{F_i(\mathbf{x},\mathbf{v})}{m} f_\sigma\right]$$

σ = p, e (proton, electron)

Implicit sum over repeated indices

Force: $\mathbf{F}(\mathbf{x},\mathbf{v}) = q_\sigma \mathbf{E}(\mathbf{x}) + q_\sigma \frac{\mathbf{v}}{c} \times \mathbf{B}(\mathbf{x})$

$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}(\mathbf{x}) = \frac{4\pi e}{c}\int \left[\mathbf{v}\, f_p(\mathbf{x},\mathbf{v}) - \mathbf{v}\, f_e(\mathbf{x},\mathbf{v})\right] d^3\mathrm{v}$ & similarly for **E**(**x**)

E(**x**) & **B**(**x**) are each linear in f_p(**x, v**) & f_e(**x, v**)

Q: what about the **pressure force**? Does that need to be included in F_i? How?

Vlasov's equation

1d version

$$\boxed{\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}\left(\mathrm{v}_x f\right) - \frac{\partial}{\partial \mathrm{v}_x}\left[\frac{F_x(x,\mathrm{v}_x)}{m} f\right]}$$

3d version

$$\frac{\partial f_\sigma}{\partial t} = -\sum_i \frac{\partial}{\partial x_i}\left(\mathrm{v}_i f_\sigma\right) - \sum_i \frac{\partial}{\partial \mathrm{v}_i}\left[\frac{F_i(\mathbf{x},\mathbf{v})}{m} f_\sigma\right] = -\frac{\partial}{\partial x_i}\left(\mathrm{v}_i f_\sigma\right) - \frac{\partial}{\partial \mathrm{v}_i}\left[\frac{F_i(\mathbf{x},\mathbf{v})}{m} f_\sigma\right]$$

σ = p, e (proton/electron)

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& similarly for **E**(**x**)

E(**x**) & **B**(**x**) are each linear in f_p(**x, v**) & f_e(**x, v**)

➔ Vlasov equations for f_p(**x, v**) & f_e(**x, v**)

- **nonlinear** in f_p(**x, v**) & f_e(**x, v**)

Vlasov's equation

$$\boxed{\frac{\partial f_\sigma}{\partial t}+\frac{\partial}{\partial x_i}\left(\mathrm{v}_i f_\sigma\right)+\frac{\partial}{\partial \mathrm{v}_i}\left[\frac{F_i(\mathbf{x},\mathbf{v})}{m}f_\sigma\right]=0}$$

$$\mathbf{F}(\mathbf{x},\mathbf{v})=q_\sigma\mathbf{E}(\mathbf{x})+q_\sigma\frac{\mathbf{v}}{c}\times\mathbf{B}(\mathbf{x})$$

f_σ(**x, v**) = **smoothed** phase space density – does not reflect discrete particles – represents an **ensemble average**

➔ **E**(**x**) & **B**(**x**) are smooth/ensemble averaged fields – not fields from discrete particles. Particles at ~same point in space feel ~same force

<u>What is **not** included:</u>

Forces between particles @ ~same point – i.e. **collisions**

Vlasov eqs. = eqs. for a **collisionless** plasma

Including Collisions: Fokker-Planck

$$\frac{\partial f_\sigma}{\partial t}+\frac{\partial}{\partial x_i}\left(\mathrm{v}_i f_\sigma\right)+\frac{\partial}{\partial \mathrm{v}_i}\left[\frac{F_i(\mathbf{x},\mathbf{v})}{m}f_\sigma\right]=\left(\frac{\partial f_\sigma}{\partial t}\right)_{\mathrm{col}}=-\frac{\partial}{\partial \mathrm{v}_i}\left(\left\langle\frac{\Delta \mathrm{v}_i}{\Delta t}\right\rangle f_\sigma\right)+\frac{1}{2}\frac{\partial^2}{\partial \mathrm{v}_i\partial \mathrm{v}_j}\left(\left\langle\frac{\Delta \mathrm{v}_i\Delta \mathrm{v}_j}{\Delta t}\right\rangle f_\sigma\right)$$

$\left\langle\frac{\Delta \mathrm{v}_i}{\Delta t}\right\rangle$ & $\left\langle\frac{\Delta \mathrm{v}_i\Delta \mathrm{v}_j}{\Delta t}\right\rangle$ are ensemble-average changes due to fluctuating **E** & **B** fields.*
Each depends on **v** – velocity
@ t – Δt prior to fluctuating forces

dependence captured in functions

$$D_{ij}(\mathbf{v})=\frac{1}{2}\left\langle\frac{\Delta \mathrm{v}_i\Delta \mathrm{v}_j}{\Delta t}\right\rangle \qquad A_i(\mathbf{v})=\left\langle\frac{\Delta \mathrm{v}_i}{\Delta t}\right\rangle-\frac{\partial D_{ij}}{\partial \mathrm{v}_j}$$

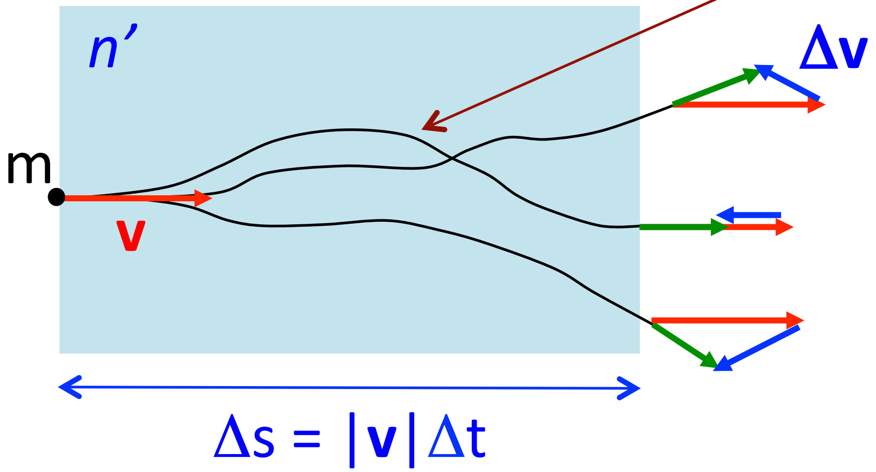
$$\boxed{\frac{\partial f_\sigma}{\partial t}+\frac{\partial}{\partial x_i}\left(\mathrm{v}_i f_\sigma\right)+\frac{\partial}{\partial \mathrm{v}_i}\left[\frac{F_i}{m}f_\sigma\right]=\left(\frac{\partial f_\sigma}{\partial t}\right)_{\mathrm{col}}=-\frac{\partial}{\partial \mathrm{v}_i}\left[A_i(\mathbf{v})f_\sigma\right]+\frac{\partial}{\partial \mathrm{v}_i}\left[D_{ij}(\mathbf{v})\frac{\partial f_\sigma}{\partial \mathrm{v}_j}\right]}$$

Fokker-Planck equation

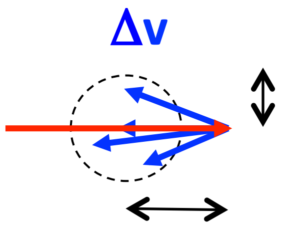
*same eqs. used for stellar dynamics: “collision” from gravitational interaction

scatterers of mass: m',
r.m.s. velocity: $v'_{rms} \ll |\mathbf{v}|$

many small-angle deflections – add up to random change $\Delta\mathbf{v}$



random walk:
displacement ~ $t^{1/2}$



$$\sqrt{\frac{8\pi e^4\,\Lambda\, n' v'^2_{rms}}{m^2 v^4}\Delta s}$$

$$\frac{8\pi e^4\,\Lambda\, n'}{m\, m'}\frac{1}{v^3}\Delta s$$ mean change

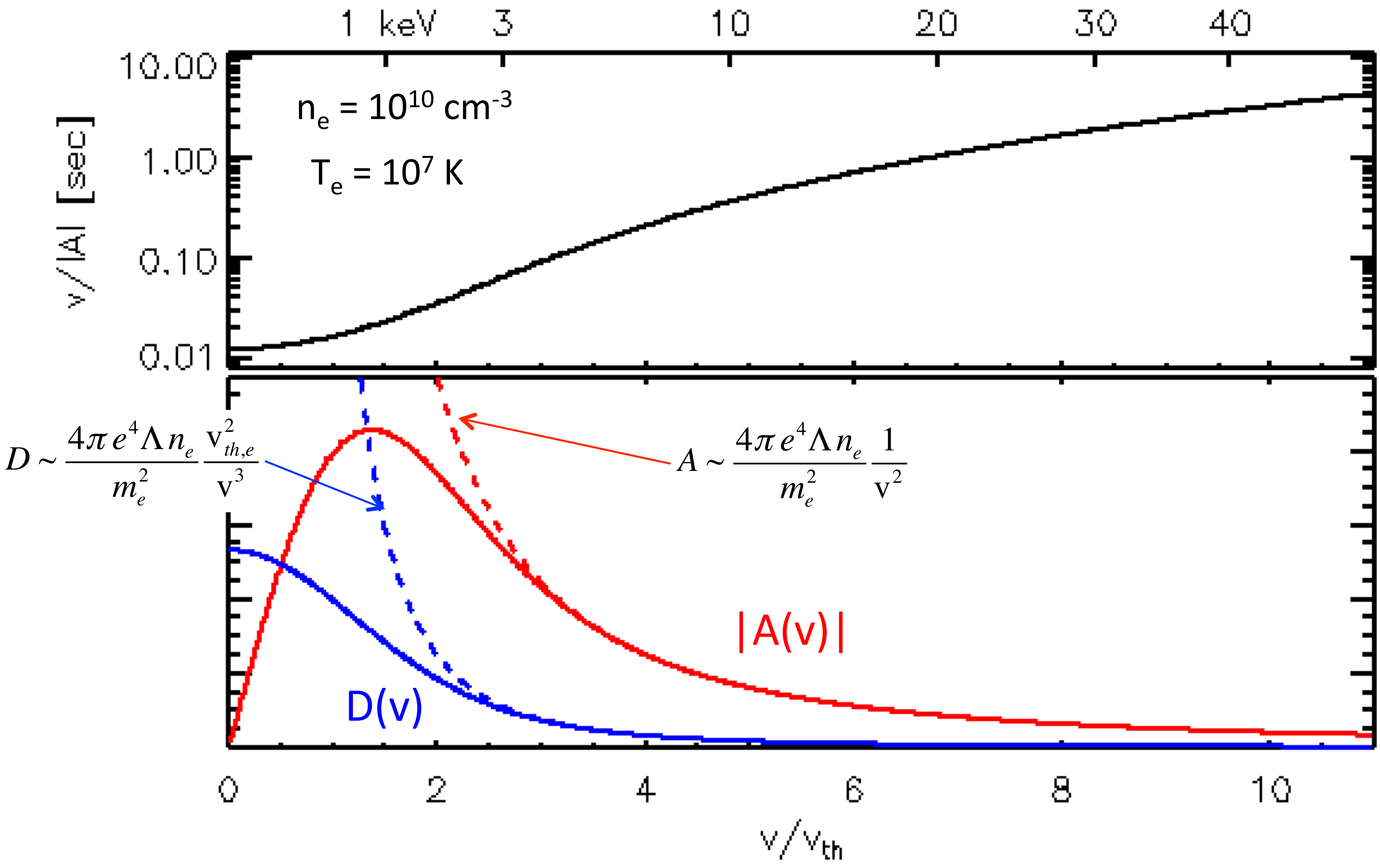
$$\left\langle \frac{\Delta v_i}{\Delta t} \right\rangle = -\frac{8\pi\, e^4 \Lambda\, n'}{m\, m'}\frac{v_i}{v^3} \qquad \left\langle \frac{\Delta v_i \Delta v_j}{\Delta t} \right\rangle = \frac{8\pi e^4\,\Lambda\, n' v'^2_{rms}}{m^2}\frac{\delta_{ij}}{v^3}$$

$$\boxed{A_i(\mathbf{v}) = -\frac{4\pi\, e^4 \Lambda\, n'}{m\, m'}\frac{v_i}{v^3}} \qquad \boxed{D_{ij}(\mathbf{v}) = \frac{4\pi e^4\,\Lambda\, n' v'^2_{rms}}{m^2}\frac{\delta_{ij}}{v^3}}$$

each is linear in $f_{\sigma'}(\mathbf{x}, \mathbf{v})$ – σ' = target species

Coulomb log Λ enters here – how many scatterers to include?

e^- / e^- collisions ($\sigma = \sigma' = e$) w/ $f_e(\mathbf{v})$ Maxwellian:



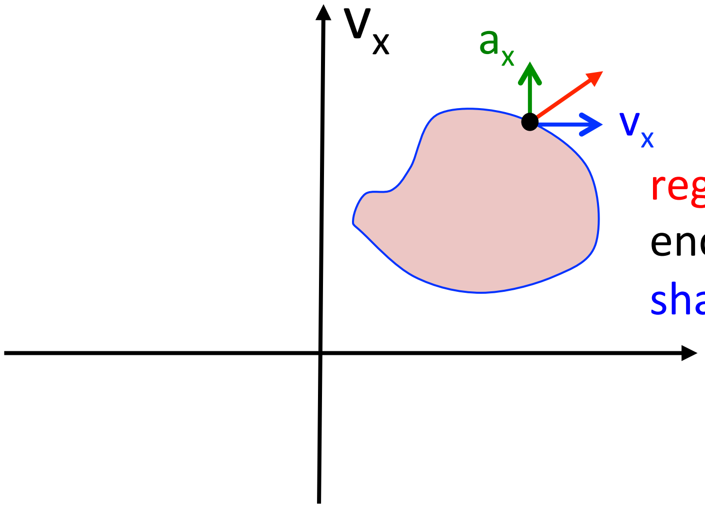
$$D \sim \frac{4\pi e^4 \Lambda n_e}{m_e^2} \frac{v_{th,e}^2}{v^3}$$

$$A \sim \frac{4\pi e^4 \Lambda n_e}{m_e^2} \frac{1}{v^2}$$

What it really means

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x_i}\left(\underset{\text{velocity}}{\mathrm{v}_i} f\right) + \frac{\partial}{\partial \mathrm{v}_i}\left[\underbrace{\left(\frac{F_i}{m} + A_i\right)}_{\text{acceleration } a_i} f\right] = \cancel{\frac{\partial}{\partial \mathrm{v}_i}\left[D_{ij}(\mathbf{v})\frac{\partial f}{\partial \mathrm{v}_j}\right]} = 0$$

what happens w/o this term?



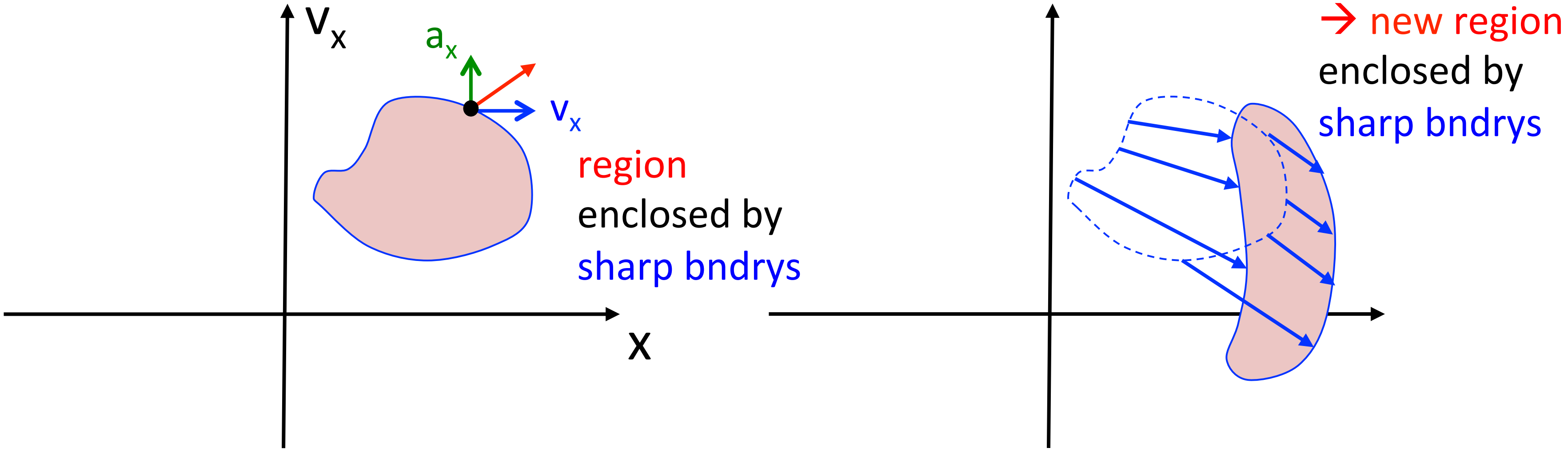
region
enclosed by
sharp bndrys

What it really means

$$\frac{\partial f}{\partial t}+\frac{\partial}{\partial x_i}\left(\mathrm{v}_i f\right)+\frac{\partial}{\partial \mathrm{v}_i}\left[\underbrace{\left(\frac{F_i}{m}+A_i\right)}_{\text{acceleration } a_i} f\right]=\cancel{\frac{\partial}{\partial \mathrm{v}_i}\left[D_{ij}(\mathbf{v})\frac{\partial f}{\partial \mathrm{v}_j}\right]}\quad = 0$$

velocity

what happens w/o this term?



region enclosed by sharp bndrys

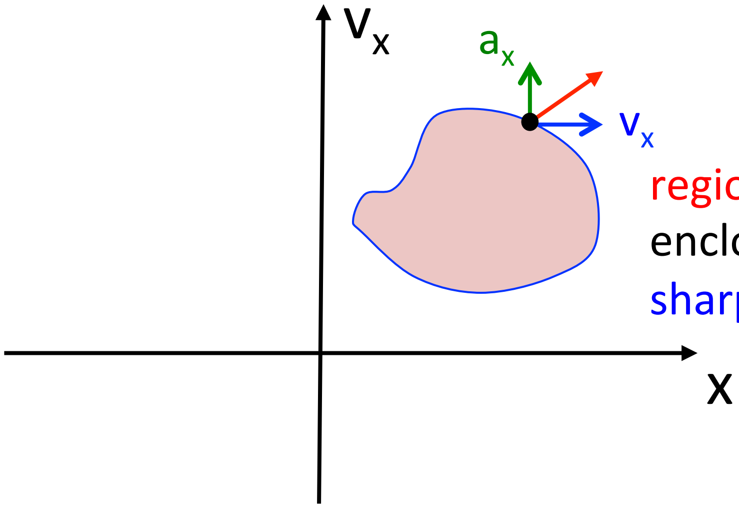
→ new region enclosed by sharp bndrys

pure advection in phase space: evolution maps

sharp bndrys → sharp bndrys

What it really means

$$\frac{\partial f}{\partial t} + \underbrace{\frac{\partial}{\partial x_i}\left(\mathrm{v}_i f\right) + \frac{\partial}{\partial \mathrm{v}_i}\left[\left(\frac{F_i}{m} + A_i\right) f\right]}_{\text{phase-space advection}} = \underbrace{\frac{\partial}{\partial \mathrm{v}_i}\left[D_{ij}(\mathbf{v}) \frac{\partial f}{\partial \mathrm{v}_j}\right]}_{\text{diffusive term}}$$



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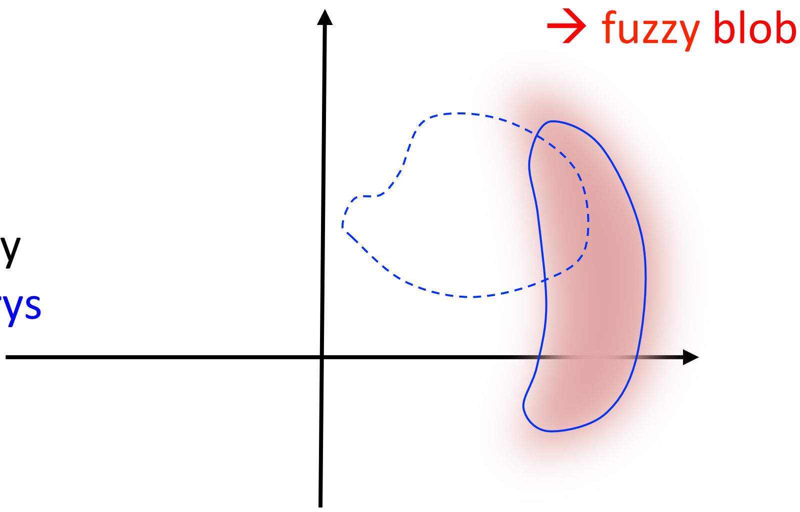


→ fuzzy blob

time irreversible!

advection/**diffusion**: D_{ij}(**v**) is **diffusion coefficient** –
diffuses only in **v** & conserves total integral = n(**x**)

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x_i}\left(\mathrm{v}_i f\right) + \frac{\partial}{\partial \mathrm{v}_i}\left[\left(\frac{F_i}{m} + A_i\right) f\right] = \frac{\partial}{\partial \mathrm{v}_i}\left[\boxed{D_{ij}(\mathbf{v})}\frac{\partial f}{\partial \mathrm{v}_j}\right]$$

$D_{ij}(\mathbf{v})$ **velocity diffusion coefficient** [$\mathrm{cm}^2/\mathrm{s}^{-3}$]

- only diffuses in **v**
- Random walk in **v** due to numerous random accelerations – i.e. random "steps" in **v**
- Compare to advection/diffusion of **spatial** field $\psi(\mathbf{x},t)$

$$\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x_i}\left[u_i(\mathbf{x})\psi(\mathbf{x},t)\right] = \frac{\partial}{\partial x_i}\left[\boxed{\kappa_{ij}(\mathbf{x})}\frac{\partial \psi}{\partial x_j}\right] \rightarrow \kappa\nabla^2\psi$$

- Diffusion coefficient $\kappa_{ij}(\mathbf{x})$ → κ [$\mathrm{cm}^2/\mathrm{s}^{-1}$]
- Random walk in **x** due to numerous random steps in **x**
- Not present in Fokker-Planck eq. – no steps in **x**

also time irreversible

What it really means

$$\left(\frac{\partial f}{\partial t}\right)_{\text{col}} = \underbrace{-\frac{\partial}{\partial \mathrm{v}_i}\left[A_i(\mathbf{v})f - D_{ij}(\mathbf{v})\frac{\partial f}{\partial \mathrm{v}_j}\right]}_{\text{velocity-space divergence}}$$

$$n(\mathbf{x}) = \int f(\mathbf{x},\mathbf{v})\,d^3\mathrm{v}$$

Q: How do collisions affect density?

$$\left(\frac{\partial n}{\partial t}\right)_{\text{col}} = \int\left(\frac{\partial f}{\partial t}\right)_{\text{col}} d^3\mathrm{v} = -\int\frac{\partial}{\partial \mathrm{v}_i}\left[A_i(\mathbf{v})f - D_{ij}(\mathbf{v})\frac{\partial f}{\partial \mathrm{v}_j}\right]d^3\mathrm{v}$$

velocity-space divergence theorem

$$= -\oint_{|\mathbf{v}|\to\infty}\left[A_i(\mathbf{v})f - D_{ij}(\mathbf{v})\frac{\partial f}{\partial \mathrm{v}_j}\right]da_i = 0$$

A: Leave it unchanged. Collisions are all at same point – change particle velocities – leave their positions **unchanged**

Q: How do collisions affect e⁻ momentum ?

$$P_{e,i} = m_e n_e u_{e,i} = m_e \int \mathrm{v}_i f_e(\mathbf{x},\mathbf{v})\, d^3\mathrm{v}$$

$$\left(\frac{\partial f_e}{\partial t}\right)_{\mathrm{col}} = -\frac{\partial}{\partial \mathrm{v}_j}\left[\left\langle\frac{\Delta \mathrm{v}_j}{\Delta t}\right\rangle f_e - \frac{1}{2}\frac{\partial}{\partial \mathrm{v}_k}\left(\left\langle\frac{\Delta \mathrm{v}_j \Delta \mathrm{v}_k}{\Delta t}\right\rangle f_e\right)\right]$$

Initial form of F-P eq.

$$\left(\frac{\partial P_{e,i}}{\partial t}\right)_{\mathrm{col}} = m_e \int \mathrm{v}_i \left(\frac{\partial f_e}{\partial t}\right)_{\mathrm{col}} d^3\mathrm{v} = -m_e \int \mathrm{v}_i \frac{\partial}{\partial \mathrm{v}_j}\left[\left\langle\frac{\Delta \mathrm{v}_j}{\Delta t}\right\rangle f_e - \frac{1}{2}\frac{\partial}{\partial \mathrm{v}_k}\left(\left\langle\frac{\Delta \mathrm{v}_j \Delta \mathrm{v}_k}{\Delta t}\right\rangle f_e\right)\right] d^3\mathrm{v}$$

velocity-space integration by parts

$$= m_e \int \underset{\delta_{ij}}{\boxed{\frac{\partial \mathrm{v}_i}{\partial \mathrm{v}_j}}}\left[\left\langle\frac{\Delta \mathrm{v}_j}{\Delta t}\right\rangle f_e - \frac{1}{2}\frac{\partial}{\partial \mathrm{v}_k}\left(\left\langle\frac{\Delta \mathrm{v}_j \Delta \mathrm{v}_k}{\Delta t}\right\rangle f_e\right)\right] d^3\mathrm{v}$$

divergence thm

$$= m_e \int \left\langle\frac{\Delta \mathrm{v}_i}{\Delta t}\right\rangle f_e d^3\mathrm{v} - \cancel{\frac{m_e}{2}\oint_{|\mathbf{v}|\to\infty}\left\langle\frac{\Delta \mathrm{v}_i \Delta \mathrm{v}_k}{\Delta t}\right\rangle f_e\, da_k}$$

A: Collisions lead to a **drag force density** on electrons

$$\boxed{\left(\frac{\partial P_{e,i}}{\partial t}\right)_{\mathrm{col}} = m_e \int \left\langle\frac{\Delta \mathrm{v}_i}{\Delta t}\right\rangle f_e d^3\mathrm{v}}$$

Collisional drag force

$$\boxed{\left(\frac{\partial P_{e,i}}{\partial t}\right)_{\text{col}} = m_e\int\left\langle\frac{\Delta \mathrm{v}_i}{\Delta t}\right\rangle f_e d^3\mathrm{v}}$$

velocity change due to collisions suffered by electrons:

$$\left\langle\frac{\Delta \mathrm{v}_i}{\Delta t}\right\rangle = \underbrace{\left\langle\frac{\Delta \mathrm{v}_i}{\Delta t}\right\rangle_{ee}}_{\text{collisions of e}^-\text{s w/ other e}^-\text{s}} + \underbrace{\left\langle\frac{\Delta \mathrm{v}_i}{\Delta t}\right\rangle_{ep}}_{\text{collisions of e}^-\text{s w/ protons}}$$

Newton's 3rd law

$$\left(\frac{\partial P_{e,i}}{\partial t}\right)_{\text{col}} = \cancel{m_e\int\left\langle\frac{\Delta \mathrm{v}_i}{\Delta t}\right\rangle_{ee}} f_e d^3\mathrm{v} + \boxed{m_e\int\left\langle\frac{\Delta \mathrm{v}_i}{\Delta t}\right\rangle_{ep} f_e d^3\mathrm{v}}$$

drag only from protons

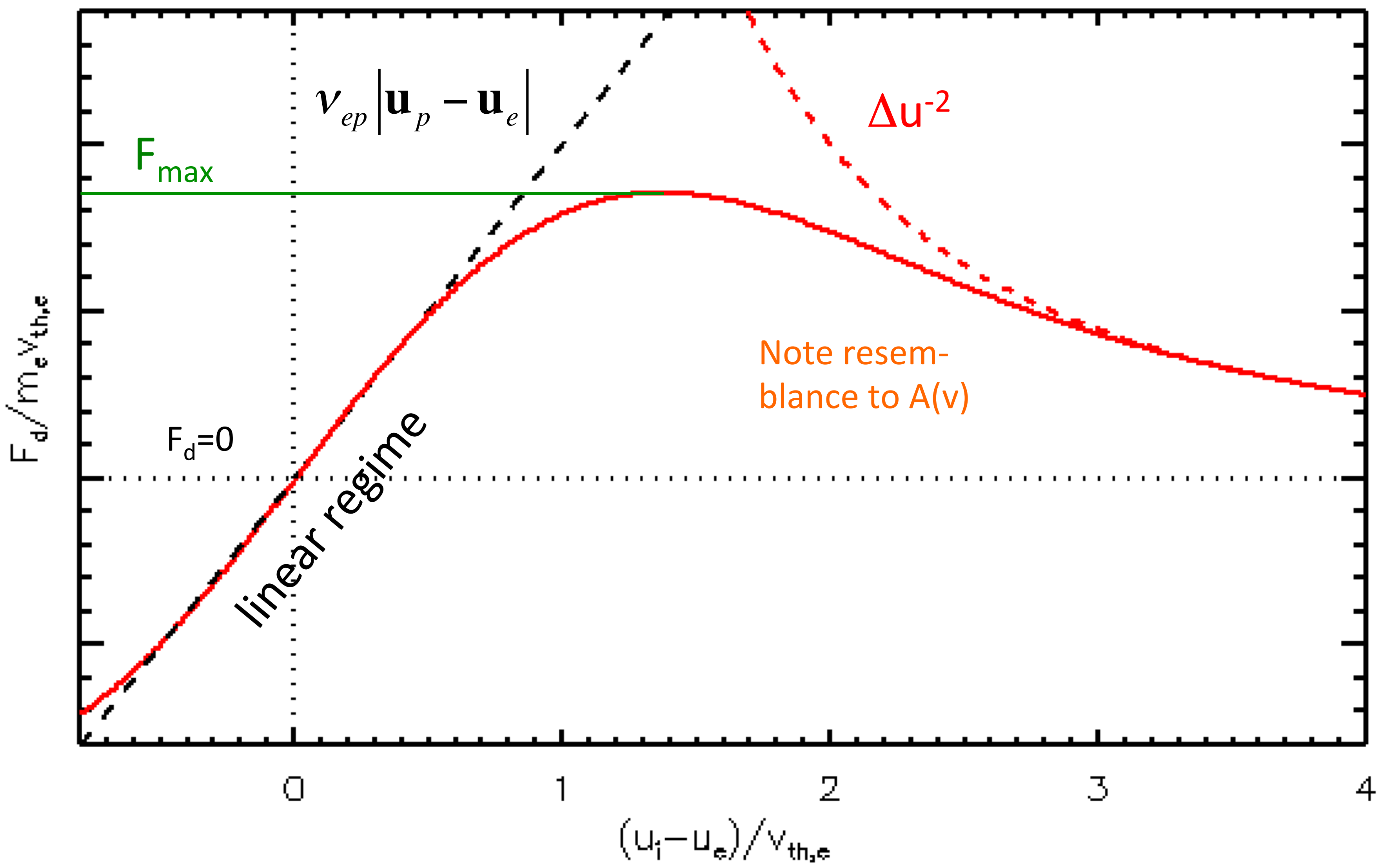
- expect drag to vanish if $\mathbf{u}_p = \mathbf{u}_e$
- lowest order in $|\mathbf{u}_p - \mathbf{u}_e| \ll v_{th,e}$

$$\boxed{m_e\int\left\langle\frac{\Delta \mathbf{v}}{\Delta t}\right\rangle_{ep} f_e d^3\mathrm{v} \approx m_e n_e \boxed{\nu_{ep}}\left(\mathbf{u}_p - \mathbf{u}_e\right) \approx \frac{m_e}{e}\nu_{ep}\,\mathbf{J}}$$

collision freq. from doing integral

$$\frac{1}{n_e}\int\left\langle\frac{\Delta \mathrm{v}_i}{\Delta t}\right\rangle_{ep} f_e(\mathbf{v})\,d^3\mathrm{v} = \frac{F_d\left(\left|\mathbf{u}_p-\mathbf{u}_e\right|\right)}{m_e}$$

Exact integral for streaming Maxwellians



Note resemblance to A(v)

Q: How do collisions affect e$^-$ energy ?

$$\varepsilon_e = \tfrac{1}{2} m_e \int v^2 f_e(\mathbf{x},\mathbf{v})\, d^3v$$

$$\left(\frac{\partial f_e}{\partial t}\right)_{\rm col} = -\frac{\partial}{\partial v_j}\left[\left\langle\frac{\Delta v_j}{\Delta t}\right\rangle f_e - \frac{1}{2}\frac{\partial}{\partial v_k}\left(\left\langle\frac{\Delta v_j \Delta v_k}{\Delta t}\right\rangle f_e\right)\right]$$

Initial form of F-P eq.

$$\left(\frac{\partial \varepsilon_e}{\partial t}\right)_{\rm col} = \tfrac{1}{2} m_e \int v^2 \left(\frac{\partial f_e}{\partial t}\right)_{\rm col} d^3v = -\tfrac{1}{2} m_e \int v^2 \frac{\partial}{\partial v_j}\left[\left\langle\frac{\Delta v_j}{\Delta t}\right\rangle f_e - \frac{1}{2}\frac{\partial}{\partial v_k}\left(\left\langle\frac{\Delta v_j \Delta v_k}{\Delta t}\right\rangle f_e\right)\right] d^3v$$

velocity-space integration by parts

$$= m_e \int v_i \underset{\delta_{ij}}{\boxed{\frac{\partial v_i}{\partial v_j}}}\left[\left\langle\frac{\Delta v_j}{\Delta t}\right\rangle f_e - \frac{1}{2}\frac{\partial}{\partial v_k}\left(\left\langle\frac{\Delta v_j \Delta v_k}{\Delta t}\right\rangle f_e\right)\right] d^3v$$

$$= \int \underbrace{m_e\left[v_j\left\langle\frac{\Delta v_j}{\Delta t}\right\rangle + \frac{1}{2}\left\langle\frac{\Delta v_j \Delta v_j}{\Delta t}\right\rangle\right]}_{L_e(\mathbf{v})} f_e\, d^3v$$

$$\boxed{\left(\frac{\partial \varepsilon}{\partial t}\right)_{\rm col} = \int L_e(\mathbf{v}) f_e d^3v}$$

Energy conservation

$$\boxed{\left(\frac{\partial \varepsilon_e}{\partial t}\right)_{\text{col}} = \int L_e(\mathbf{v})\, f_e\, d^3\text{v}}$$

$$L_e(\mathbf{v}) = \underbrace{m_e\left(\text{v}_i\left\langle\frac{\Delta\text{v}_i}{\Delta t}\right\rangle_{ee} + \frac{1}{2}\left\langle\frac{\Delta\text{v}_i\Delta\text{v}_i}{\Delta t}\right\rangle_{ee}\right)}_{L_{ee}(\mathbf{v})\text{: collisions of } e^-\text{s w/ other } e^-\text{s}} + \underbrace{m_e\left(\text{v}_i\left\langle\frac{\Delta\text{v}_i}{\Delta t}\right\rangle_{ep} + \frac{1}{2}\left\langle\frac{\Delta\text{v}_i\Delta\text{v}_i}{\Delta t}\right\rangle_{ep}\right)}_{L_{ep}(\mathbf{v})\text{: collisions of } e^-\text{s w/ protons}}$$

$$\text{Elastic collisions:}\quad \left\{\begin{array}{l} \int L_{ee}(\mathbf{v})\, f_e\, d^3\text{v} = 0 \\[2ex] \int L_{ep}(\mathbf{v})\, f_e\, d^3\text{v} + \int L_{pe}(\mathbf{v})\, f_p\, d^3\text{v} = 0 \end{array}\right.$$

energy exchange between e$^-$s & protons

Q: Why might the collisions be inelastic?
What would that do to the above relations?

What does $D_{ij}(\mathbf{v})$ do?

$$\left(\frac{\partial f}{\partial t}\right)_{\text{col}} = -\frac{\partial}{\partial \mathrm{v}_i}\left[A_i(\mathbf{v})f\right] + \frac{\partial}{\partial \mathrm{v}_i}\left[D_{ij}(\mathbf{v})\frac{\partial f}{\partial \mathrm{v}_j}\right] = -\frac{\partial}{\partial \mathrm{v}_i}\left\{A_i(\mathbf{v})\left[f - \frac{D_{ij}(\mathbf{v})}{A_i(\mathbf{v})}\frac{\partial f}{\partial \mathrm{v}_j}\right]\right\}$$

$$A_i(\mathbf{v}) = -\frac{4\pi e^4 \Lambda n}{m^2}\frac{\mathrm{v}_i}{\mathrm{v}^3} \qquad D_{ij}(\mathbf{v}) = \frac{4\pi e^4 \Lambda n \mathrm{v}_{rms}^2}{m^2}\frac{\delta_{ij}}{\mathrm{v}^3} \qquad \boxed{\frac{D_{ij}(\mathbf{v})}{A_i(\mathbf{v})} = -\frac{\mathrm{v}_{rms}^2}{\mathrm{v}_i}\delta_{ij}}$$

$$f(\mathbf{v}) \propto \exp\left(-\frac{|\mathbf{v}|^2}{2\mathrm{v}_{\text{rms}}^2}\right) \Rightarrow \frac{\partial f}{\partial \mathrm{v}_j} = -\frac{\mathrm{v}_j}{\mathrm{v}_{\text{rms}}^2} f \quad \rightarrow \quad \left(\frac{\partial f}{\partial t}\right)_{\text{col}} = 0$$

A: It enforces 2nd law of thermodynamics:

- Maxwellian is steady solution ($\mathrm{v}_{\mathrm{rms}} = \mathrm{v}_{\mathrm{th}}$)
- Collisions do not change a Maxwellian (A_i & D_{ij} cancel out)
- D_{ij} relaxes (diffuses) $f(\mathbf{v})$ toward a Maxwellian

$$A_i(\mathbf{v}) = -\frac{4\pi e^4 \Lambda n'}{m m'}\frac{\mathrm{v}_i}{\mathrm{v}^3} \qquad D_{ij}(\mathbf{v}) = \frac{4\pi e^4 \Lambda n' \mathrm{v}'^2_{rms}}{m^2}\frac{\delta_{ij}}{\mathrm{v}^3} \qquad \mathrm{v} \gg \mathrm{v}_{th}$$

each is linear in $f_{\sigma'}(\mathbf{x}, \mathbf{v})$ of target species σ'

➔ decompsing into thermal & non-thermal components

$$f_{\sigma'}(\mathbf{x}, \mathbf{v}) = f^{(th)}_{\sigma'}(\mathbf{x}, \mathbf{v}) + f^{(nt)}_{\sigma'}(\mathbf{x}, \mathbf{v}) \quad , \quad n_{\sigma'}^{(th)} \gg n_{\sigma'}^{(nt)}$$

➔ $A_i(\mathbf{v}) = A_i^{(th)}(\mathbf{v}) + \cancel{A_i^{(nt)}(\mathbf{v})}$ & similar for $D_{ij}(\mathbf{v})$

$$\left(\frac{\partial f}{\partial t}\right)_{\mathrm{col}} \approx -\frac{\partial}{\partial \mathrm{v}_i}\left\{\left[A_i^{(th)}(\mathbf{v}) - D_{ij}^{(th)}(\mathbf{v})\frac{\partial}{\partial \mathrm{v}_j}\right]\left(f^{(th)} + f^{(nt)}\right)\right\} \approx -\frac{\partial}{\partial \mathrm{v}_i}\left\{\left[A_i^{(th)}(\mathbf{v}) - D_{ij}^{(th)}(\mathbf{v})\frac{\partial}{\partial \mathrm{v}_j}\right] f^{(nt)}\right\}$$

$$\left(\frac{\partial f}{\partial t}\right)_{\mathrm{col}} = 0$$

when both are
Maxwellians of same T

$$A_i(\mathbf{v}) = -\frac{4\pi e^4 \Lambda n'}{m m'}\frac{v_i}{v^3} \qquad D_{ij}(\mathbf{v}) = \frac{4\pi e^4 \Lambda n' v'^2_{rms}}{m^2}\frac{\delta_{ij}}{v^3} \qquad v >> v_{th}$$

each is linear in $f_{\sigma'}(\mathbf{x}, \mathbf{v})$ of target species σ'

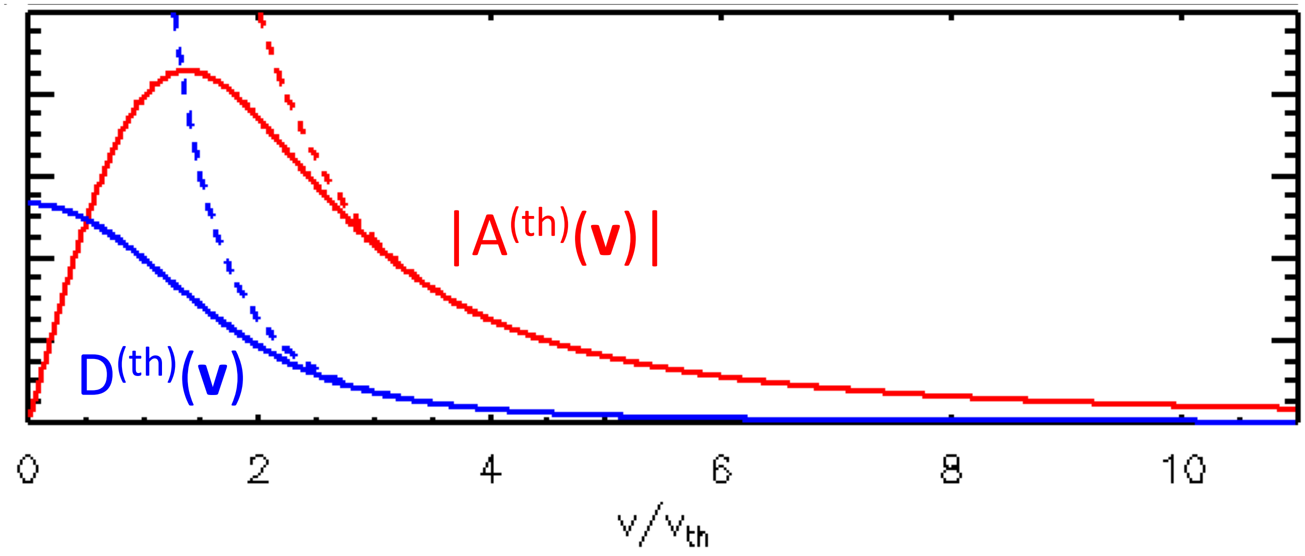
➔ decompsing into thermal & non-thermal components

$f_{\sigma'}(\mathbf{x}, \mathbf{v}) = f^{(th)}{}_{\sigma'}(\mathbf{x}, \mathbf{v}) + f^{(nt)}{}_{\sigma'}(\mathbf{x}, \mathbf{v})$, $n_{\sigma'}{}^{(th)} >> n_{\sigma'}{}^{(nt)}$

➔ $A_i(\mathbf{v}) = A_i^{(th)}(\mathbf{v}) + \cancel{A_i^{(nt)}(\mathbf{v})}$ & similar for $D_{ij}(\mathbf{v})$

$$\left(\frac{\partial f}{\partial t}\right)_{\mathrm{col}} \approx -\frac{\partial}{\partial v_i}\left\{\left[A_i^{(th)}(\mathbf{v}) - D_{ij}^{(th)}(\mathbf{v})\frac{\partial}{\partial v_j}\right]\left(f^{(th)} + f^{(nt)}\right)\right\} \approx -\frac{\partial}{\partial v_i}\left\{\left[A_i^{(th)}(\mathbf{v}) - D_{ij}^{(th)}(\mathbf{v})\frac{\partial}{\partial v_j}\right]f^{(nt)}\right\}$$

➔ can use the collision coefficients computed w/ Maxwellian distribution



Electron momentum eq. – *a.k.a.* Ohm’s law

$$\boxed{\frac{\partial f_e}{\partial t} = -\frac{\partial}{\partial x_j}\left(\mathrm{v}_j f_e\right) - \frac{\partial}{\partial \mathrm{v}_j}\left(\frac{F_j}{m_e} f_e\right) + \left(\frac{\partial f_e}{\partial t}\right)_{\mathrm{col}}}$$

$$\frac{\partial}{\partial t}\left(m_e n_e u_{e,i}\right) = \int m_e \mathrm{v}_i \boxed{\frac{\partial f_e}{\partial t}} d^3\mathrm{v} = -\frac{\partial}{\partial x_j}\underbrace{\int m_e \mathrm{v}_i \mathrm{v}_j f_e d^3\mathrm{v}} - \int \mathrm{v}_i \frac{\partial}{\partial \mathrm{v}_j}\left(F_j f_e\right) d^3\mathrm{v}$$

$$+ m_e \int \mathrm{v}_i \left(\frac{\partial f_e}{\partial t}\right)_{\mathrm{col}} d^3\mathrm{v}$$

$$\boxed{m_e \int \left(\mathrm{v}_i - u_i\right)\left(\mathrm{v}_j - u_j\right) f_e d^3\mathrm{v} + m_e \int u_i u_j f_e d^3\mathrm{v} = p_{ij} + m_e n_e u_i u_j}$$

Electron momentum eq. – *a.k.a.* Ohm's law

$$\boxed{\frac{\partial f_e}{\partial t} = -\frac{\partial}{\partial x_j}\left(\mathrm{v}_j f_e\right) - \frac{\partial}{\partial \mathrm{v}_j}\left(\frac{F_j}{m_e} f_e\right) + \left(\frac{\partial f_e}{\partial t}\right)_{\mathrm{col}}}$$

$$\frac{\partial}{\partial t}\left(m_e n_e u_{e,i}\right) = \int m_e \mathrm{v}_i \boxed{\frac{\partial f_e}{\partial t}} d^3\mathrm{v} = -\frac{\partial}{\partial x_j}\int m_e \mathrm{v}_i \mathrm{v}_j f_e d^3\mathrm{v} \underbrace{- \int \mathrm{v}_i \frac{\partial}{\partial \mathrm{v}_j}\left(F_j f_e\right) d^3\mathrm{v}}_{} + m_e \int \mathrm{v}_i \left(\frac{\partial f_e}{\partial t}\right)_{\mathrm{col}} d^3\mathrm{v}$$

$$\boxed{m_e \int \left(\mathrm{v}_i - u_i\right)\left(\mathrm{v}_j - u_j\right) f_e d^3\mathrm{v} + m_e \int u_i u_j f_e d^3\mathrm{v} = p_{ij} + m_e n_e u_i u_j}$$

integration
by parts

$$\boxed{\int \overset{\delta_{ij}}{\boxed{\frac{\partial \mathrm{v}_i}{\partial \mathrm{v}_j}}} F_j f_e d^3\mathrm{v} = \int F_i f_e d^3\mathrm{v} = n_e \overline{F}_i = -n_e e E_i - \frac{n_e e}{c}\left(\mathbf{u}_e \times \mathbf{B}\right)_i}$$

Electron momentum eq. – *a.k.a.* Ohm's law

$$\frac{\partial}{\partial t}\left(m_e n_e u_{e,i}\right) = -\frac{\partial}{\partial x_j}\left(m_e n_e u_{e,i} u_{e,j} + p_{ij}\right) - m_e n_e e\left[E_i + \frac{1}{c}\left(\mathbf{u}_e \times \mathbf{B}\right)_i\right]$$

$$\underbrace{+\, m_e \int \mathrm{v}_i \left(\frac{\partial f_e}{\partial t}\right)_{\mathrm{col}} d^3\mathrm{v}}$$

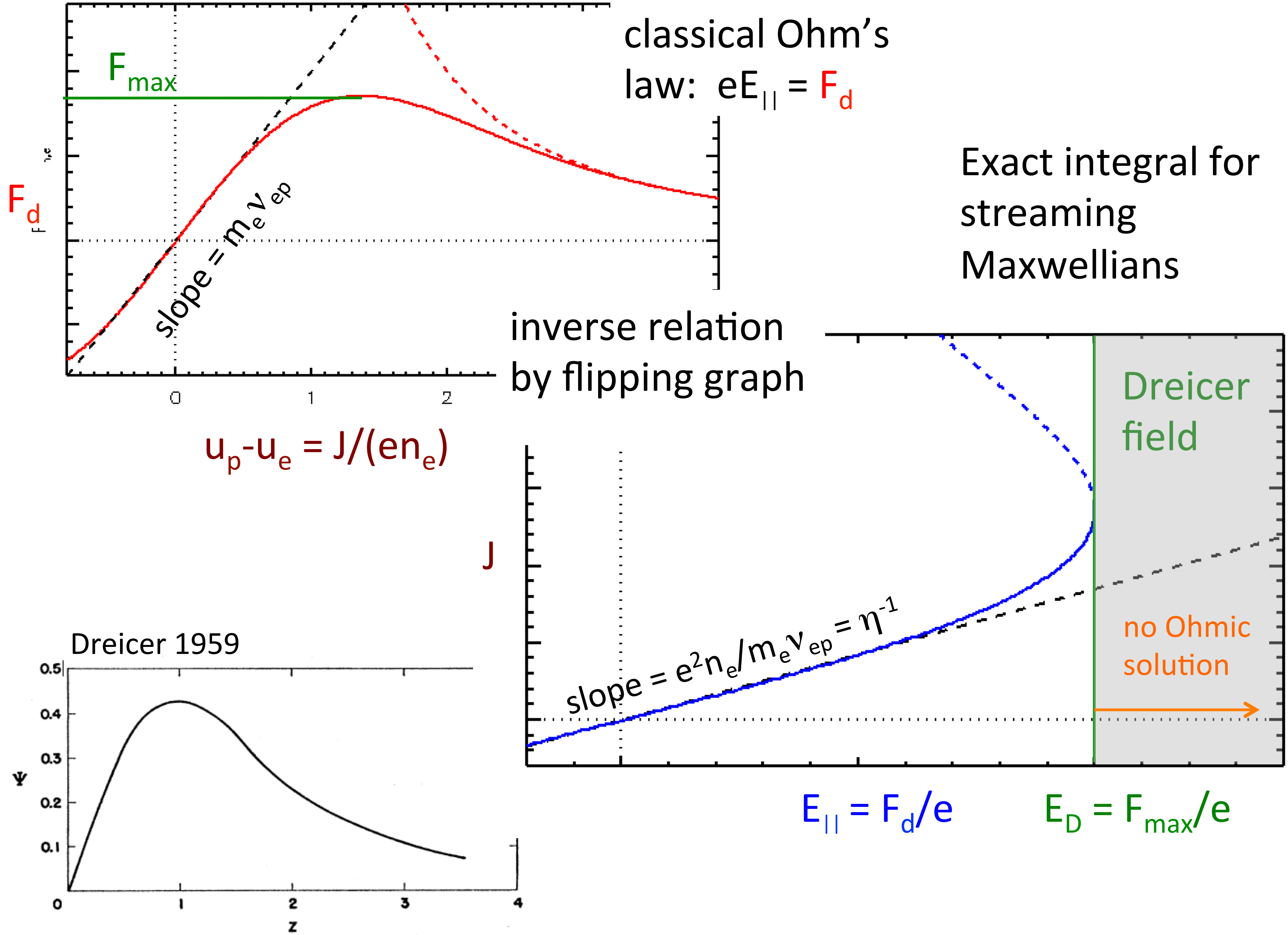
$$\boxed{m_e \int \left\langle \frac{\Delta \mathbf{v}}{\Delta t} \right\rangle_{ep} f_e d^3\mathrm{v} \approx \frac{m_e}{e} \nu_{ep} \mathbf{J}}$$

1st moment of e⁻ F-P eqn. ➔ **generalized Ohm's law**

$$\boxed{m_e n_e \frac{\partial \mathbf{u}_e}{\partial t} + m_e n_e \left(\mathbf{u}_e \cdot \nabla\right) \mathbf{u}_e = -\nabla \cdot \overleftrightarrow{p}_e - e n_e \underbrace{\left[\mathbf{E} + \frac{1}{c} \mathbf{u}_e \times \mathbf{B} - \overset{\eta}{\boxed{\frac{m_e}{e^2 n_e} \nu_{ep}}} \mathbf{J}\right]}_{=\,0 \text{ in classical Ohm's law}}}$$

e-p collision coefficient $\left\langle \frac{\Delta \mathrm{v}_i}{\Delta t} \right\rangle_{ep} \sim A_i^{(ep)}(\mathbf{v})$

→ resistivity η



classical Ohm's law: $eE_{||} = F_d$

Exact integral for streaming Maxwellians

inverse relation by flipping graph

Dreicer 1959

Ohm’s law describes 1st moment of f(**v**)… Fokker-Planck eqn. describes full function

$$\boxed{\frac{\partial f_e}{\partial t}+\frac{\partial}{\partial x_i}\left(\mathrm{v}_i f_e\right)+\frac{\partial}{\partial \mathrm{v}_i}\left(\frac{F_i}{m}f_e\right) = -\frac{\partial}{\partial \mathrm{v}_i}\left[A_i(\mathbf{v})f_e\right]+\frac{\partial}{\partial \mathrm{v}_i}\left[D_{ij}(\mathbf{v})\frac{\partial f_e}{\partial \mathrm{v}_j}\right]}$$

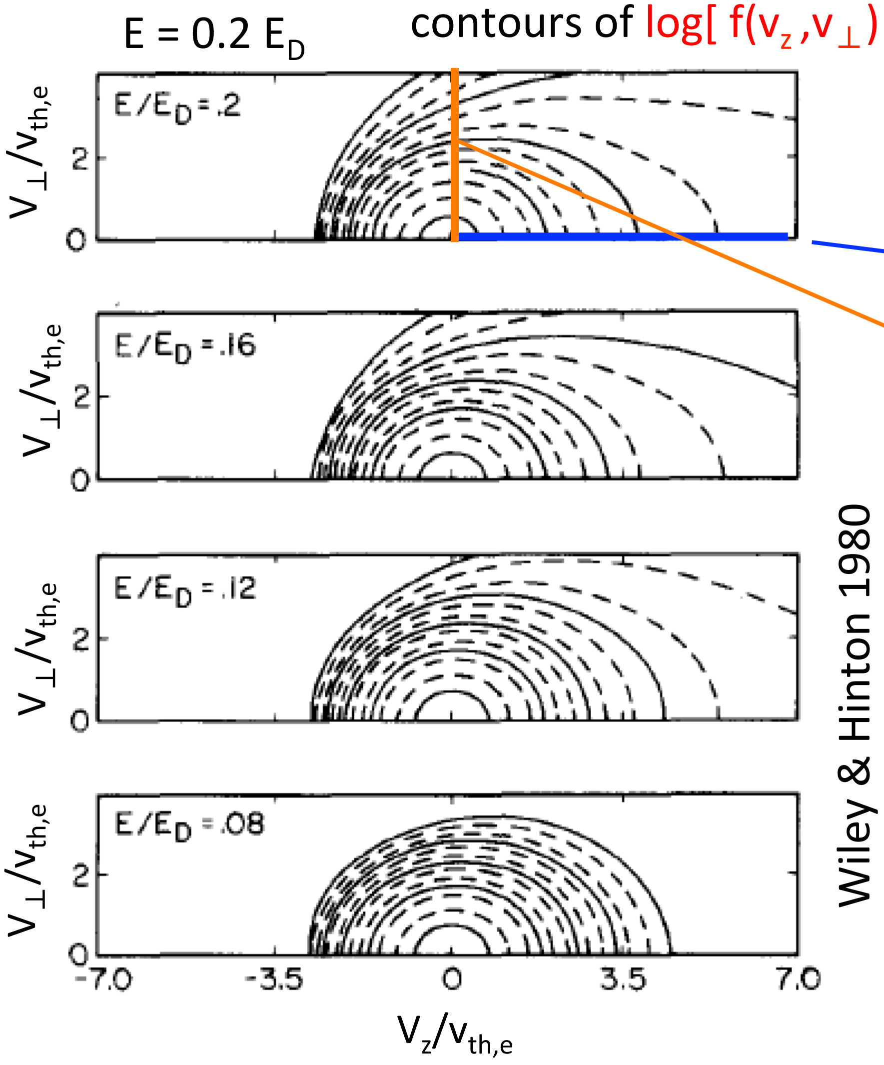
Response to E:

- idealized problem due to Spitzer & Härm (1953), Kruskal & Bernstein (1964), Kulsrud, et al. (1973)
- spatially uniform: $\nabla = 0$
- protons **retain** stationary Max’n distribution
- initialize e^-s w/ stationary uniform Max’n dist’n – $T_e = T_p$
- @ t=0 introduce uniform $\mathbf{E} = -E\,\hat{\mathbf{z}}$ w/ $E < E_D$
- solve F-P eq. for $f_e(\mathrm{v}_z, \mathrm{v}_\perp, t)$ $t > 0$

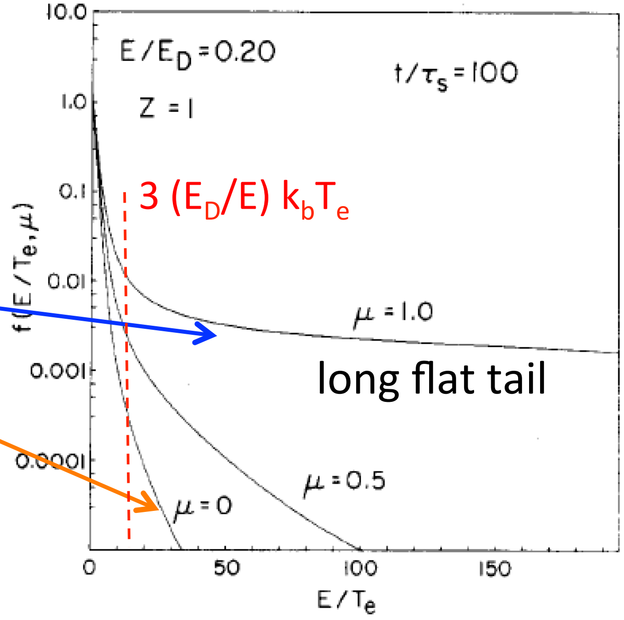
$$\boxed{\frac{\partial f_e}{\partial t}=eE\frac{\partial f_e}{\partial \mathrm{v}_z}-\frac{\partial}{\partial \mathrm{v}_i}\left[A_i(\mathbf{v})f_e\right]+\frac{\partial}{\partial \mathrm{v}_i}\left[D_{ij}(\mathbf{v})\frac{\partial f_e}{\partial \mathrm{v}_j}\right]}$$

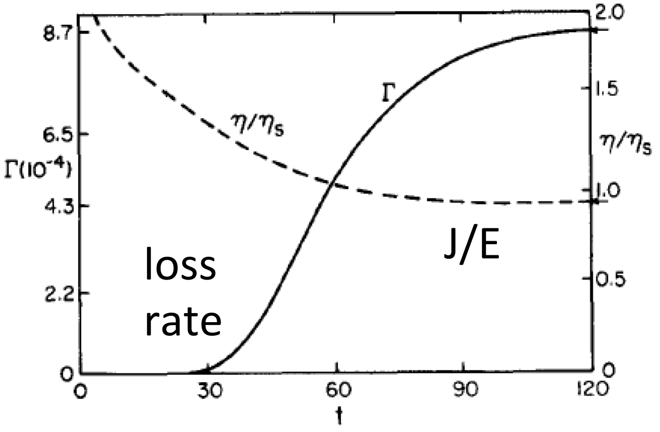
$$\boxed{\frac{\partial f_e}{\partial t} = eE\frac{\partial f_e}{\partial v_z} - \frac{\partial}{\partial v_i}\left[A_i(\mathbf{v})f_e\right] + \frac{\partial}{\partial v_i}\left[D_{ij}(\mathbf{v})\frac{\partial f_e}{\partial v_j}\right]}$$

A_i & D_{ij} from initial, Max'n, e^- & p dist'ns



Wiley & Hinton 1980

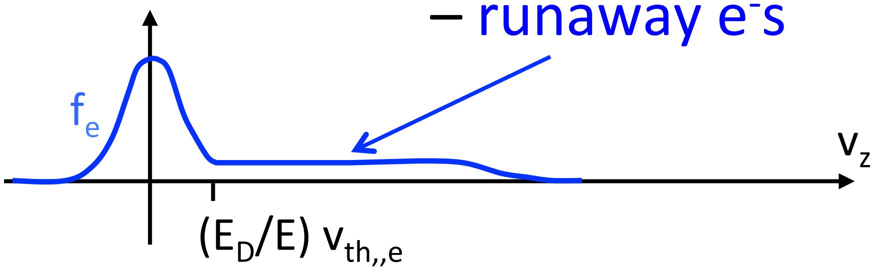




response to E:

- $f_e(v_z, v_\perp, t)$ develops long tail

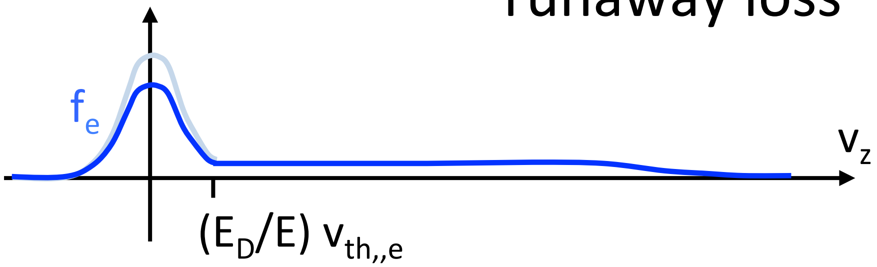
  – runaway e^-s



- tail grows in time

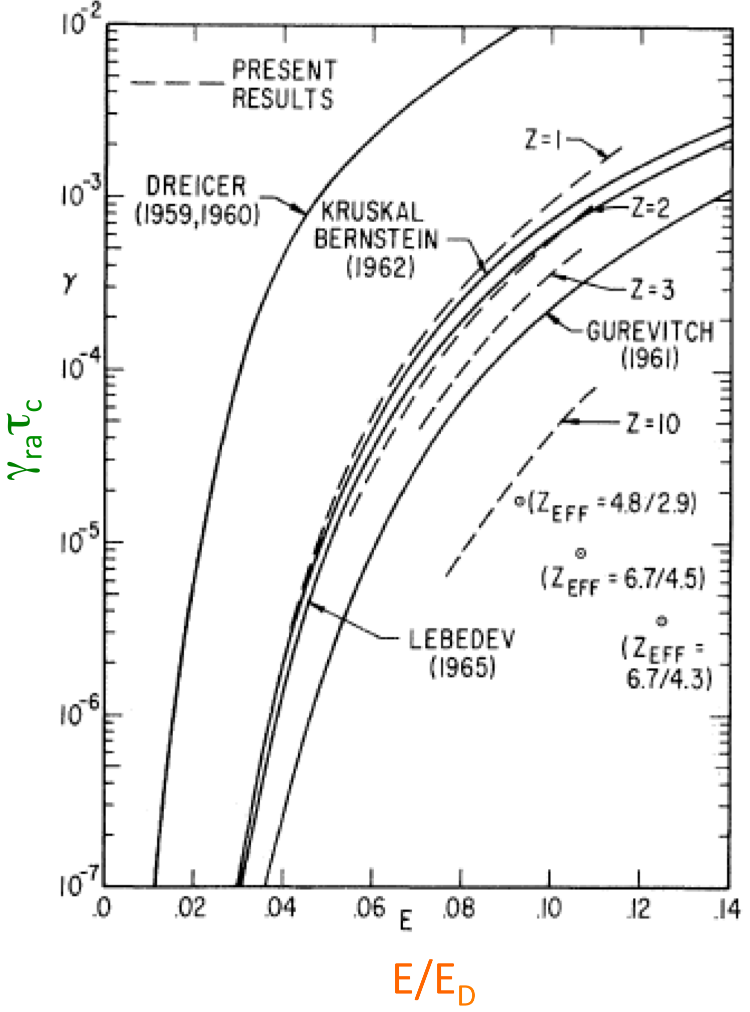
  – core decreases $\sim e^{-\gamma t}$

  runaway loss

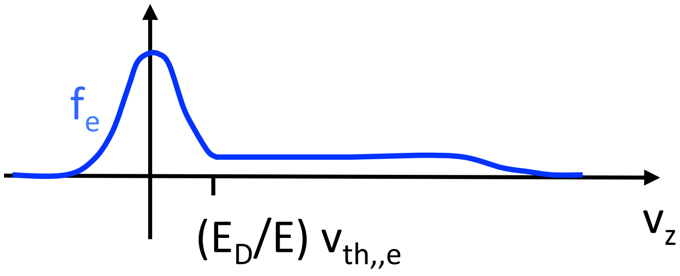


- Runaway loss rate, γ_{ra} , depends **strongly** on E/E_D

Kulsrud et al. 1973

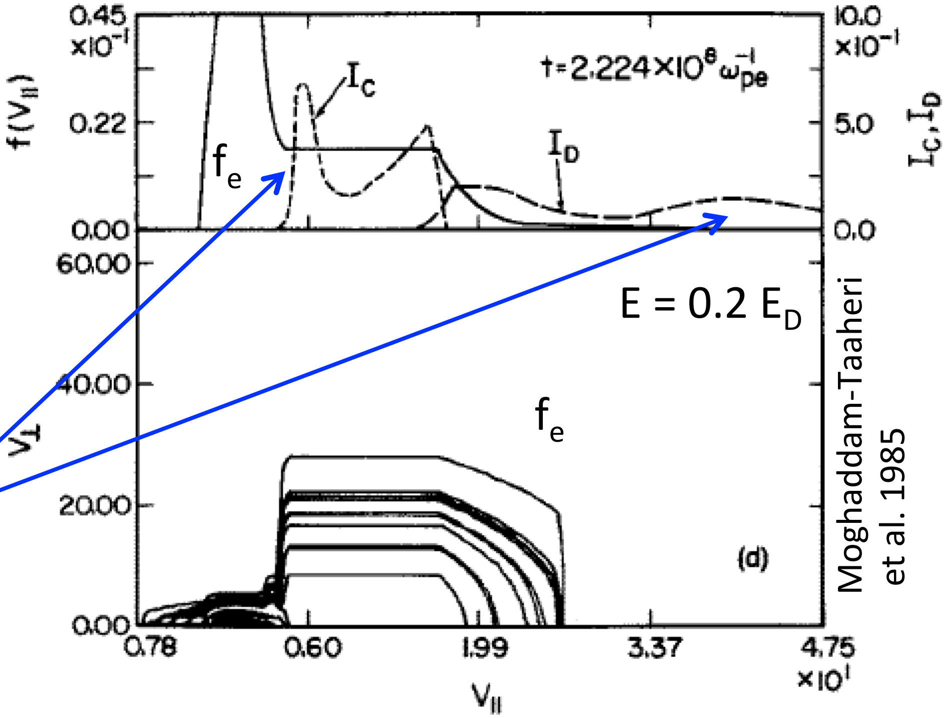


What this means for flares:



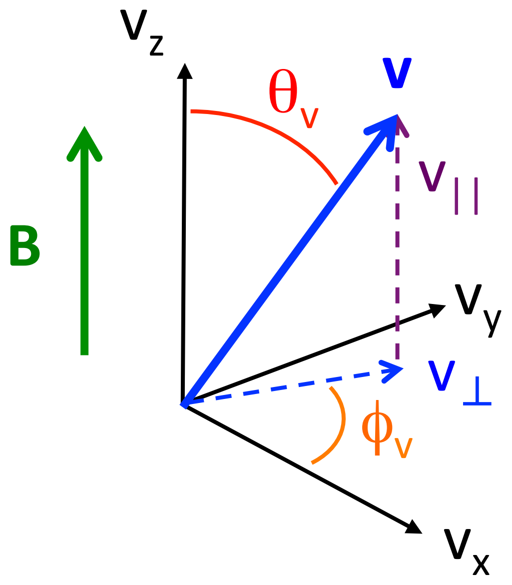
- tail is not a power law !?

  (can use in complicated model)
- $E \ll E_D$ – lest a significant fraction of e$^-$s are lost
  - would leave +ve net charge in corona
  - ➔ large opposing E – cancel out original E
  - No longer homogeneous problem – need geometry
- dist’n w/ tail: **unstable**
  - create turbulence of plasma waves (LH ⟶
  - modifies A_i & D_{ij} – more to come about this
  - all current treatments include effects of turbulent wave spectra



E = 0.2 E_D

Moghaddam-Taaheri et al. 1985



$$\mathbf{A}(\mathbf{v}) = A(\mathrm{v})\hat{\mathbf{v}} \qquad \overleftrightarrow{D}(\mathbf{v}) = D(\mathrm{v})\overleftrightarrow{I}$$

gyromotion ➔ no dep'nce on ϕ_v

$$f(\mathbf{v}) = \frac{\tilde{f}(\mathrm{v},\mu)}{2\pi \mathrm{v}^2}$$

$$\frac{\partial}{\partial \mathbf{v}} \cdot \left[\mathbf{A}(\mathbf{v}) f(\mathbf{v})\right] = \frac{1}{2\pi}\frac{1}{\mathrm{v}^2}\frac{\partial}{\partial \mathrm{v}}\left[A(\mathrm{v})\tilde{f}\right]$$

$$\frac{\partial f(\mathbf{v})}{\partial \mathbf{v}} = \frac{1}{2\pi}\left[\frac{1}{\mathrm{v}^2}\frac{\partial \tilde{f}}{\partial \mathrm{v}} - \frac{2}{\mathrm{v}^3}\tilde{f}\right]\hat{\mathbf{v}} + \frac{1}{2\pi}\frac{1}{\mathrm{v}^3}\boxed{\frac{d\mu}{d\theta_\mathrm{v}}}\frac{\partial \tilde{f}}{\partial \mu}\hat{\theta}_\mathrm{v}$$

$$\boxed{= -\sin(\theta_\mathrm{v}) = -\sqrt{1-\mu^2}}$$

$$\frac{\partial}{\partial \mathbf{v}} \cdot \left[\overleftrightarrow{D}(\mathbf{v}) \cdot \frac{\partial f(\mathbf{v})}{\partial \mathbf{v}}\right] = \frac{1}{2\pi}\frac{1}{\mathrm{v}^2}\left\{\frac{\partial}{\partial \mathrm{v}}\left[D(\mathrm{v})\left(\frac{\partial \tilde{f}}{\partial \mathrm{v}} - \frac{2}{\mathrm{v}}\tilde{f}\right)\right] + \frac{D(\mathrm{v})}{\mathrm{v}^2}\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial \tilde{f}}{\partial \mu}\right]\right\}$$

Fokker-Planck equation in (v,μ) space

$$\tilde{f}(\mathrm{v},\mu) = 2\pi \mathrm{v}^2 f(\mathbf{v})$$

$$\left(\frac{\partial \tilde{f}}{\partial t}\right)_{\mathrm{col}} = -\frac{\partial}{\partial \mathrm{v}}\left\{\left[A(\mathrm{v}) + \frac{2D(\mathrm{v})}{\mathrm{v}}\right]\tilde{f}\right\} + \underbrace{\frac{\partial}{\partial \mathrm{v}}\left[D(\mathrm{v})\frac{\partial \tilde{f}}{\partial \mathrm{v}}\right]}_{\text{energy diffusion}} + \underbrace{\frac{D(\mathrm{v})}{\mathrm{v}^2}\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial \tilde{f}}{\partial \mu}\right]}_{\text{pitch angle diffusion – works to make f uniform in pitch angle cosine}}$$

➔ distribution uniform over sphere of constant v

$$\frac{\partial \tilde{f}}{\partial \mu} \to 0$$

Summary

- Distribution function $f_\sigma(\mathbf{x},\mathbf{v})$ for species σ (=e,p) evolves by Fokker-Planck (F-P) equation
- F-P eqn. includes collisions via coefficients $A_i(\mathbf{v})$ and $D_{ij}(\mathbf{v})$
- Account for fluid effects include resistivity
- F-P shows how $f_e(\mathbf{x},\mathbf{v})$ responds to DC **E**

<u>Next:</u> How F-P can account for plasma waves – **Stochastic Acceleration**