Non-thermal particles

The distribution function

Lecture 17

March 27, 2017

Last time: How does 1 charged particle evolve?

A: **x** and **v** change in time – for particle on field line use s, v and μ

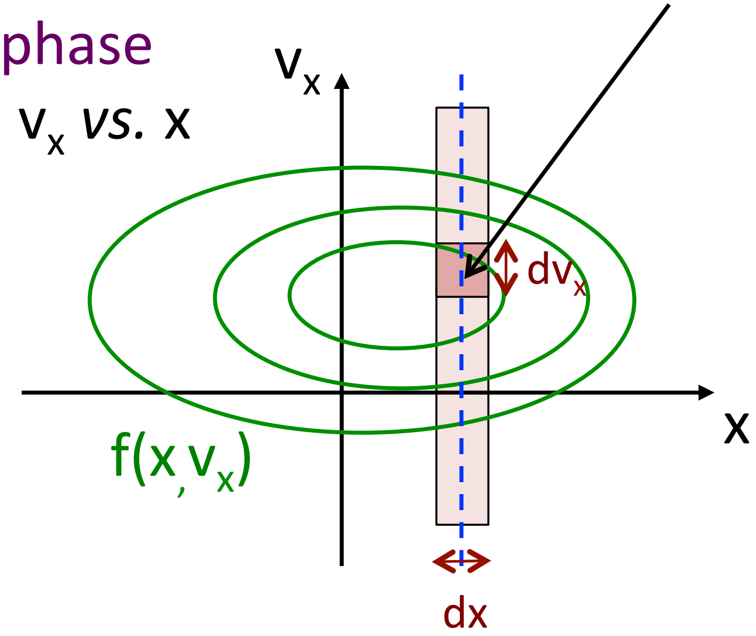
This time: How do we describe a collection of charged particles? (i.e. a plasma)

A: with a distribution function f(s,v,μ)

Next time: How does the particle evolution produce an evolution of f(s,v,μ)?

A: by the Fokker-Planck equation

The distribution function



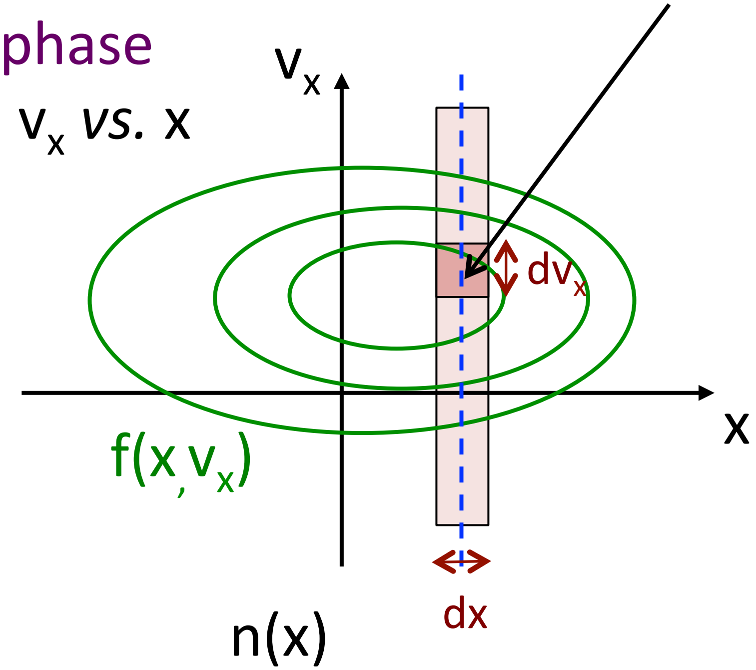
particles in phase-space volume: $f(x,v_x)\ dx\ dv_x$

kinda-phase space: v_x *vs.* x

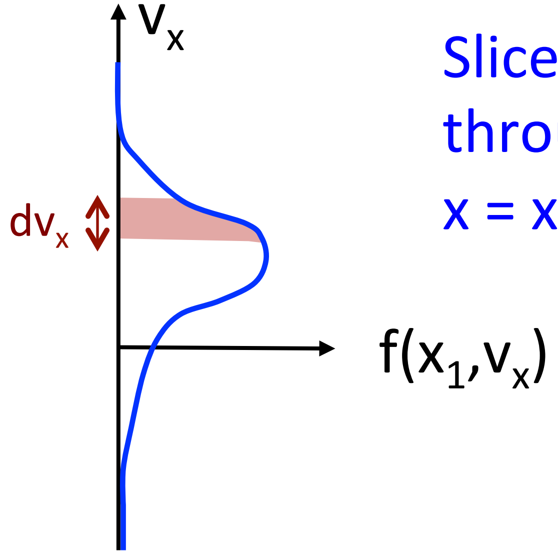
The distribution function

particles in phase-space volume: $f(x,v_x)\,dx\,dv_x$

kinda-phase space: v_x *vs.* x



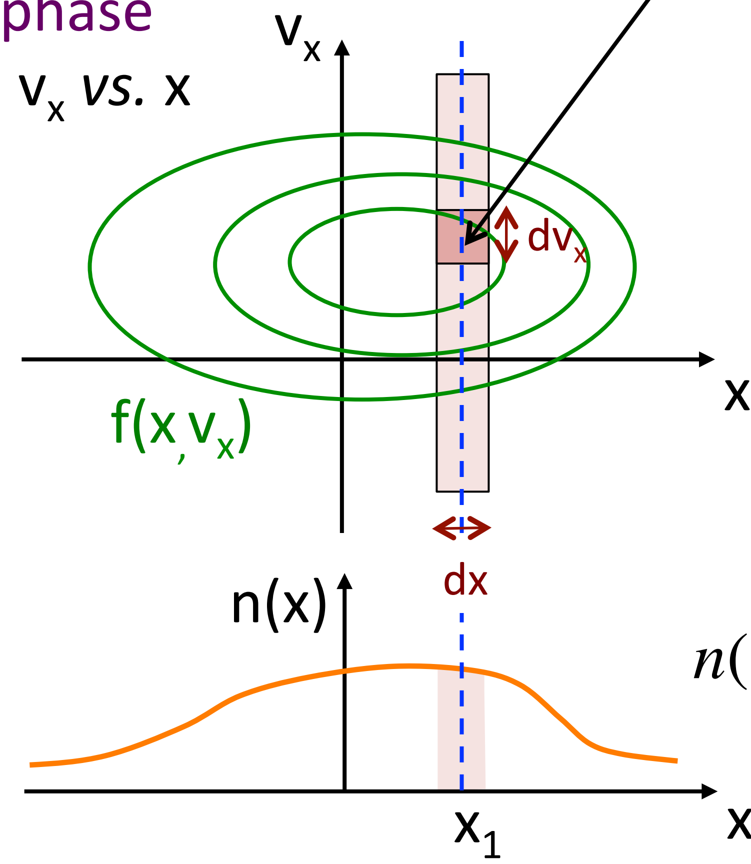
Slice through $x = x_1$

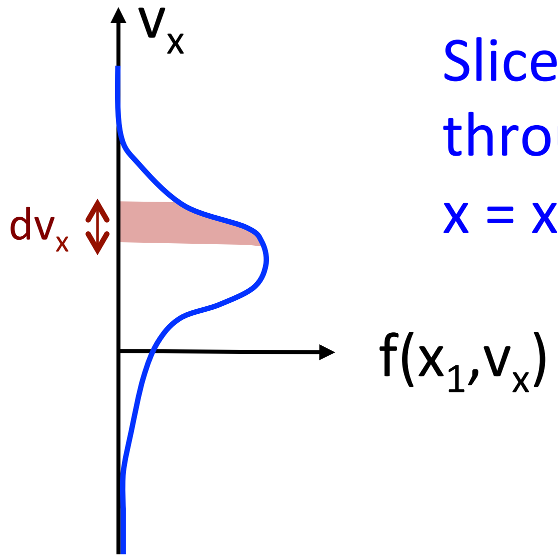


The distribution function

particles in phase-space volume: $f(x, v_x)\, dx\, dv_x$

kinda-phase space: v_x *vs.* x





Slice through $x = x_1$

$$n(x) = \int_{-\infty}^{\infty} f(x, \mathrm{v}_x)\, d\mathrm{v}_x$$

number density: integrate over all* velocities

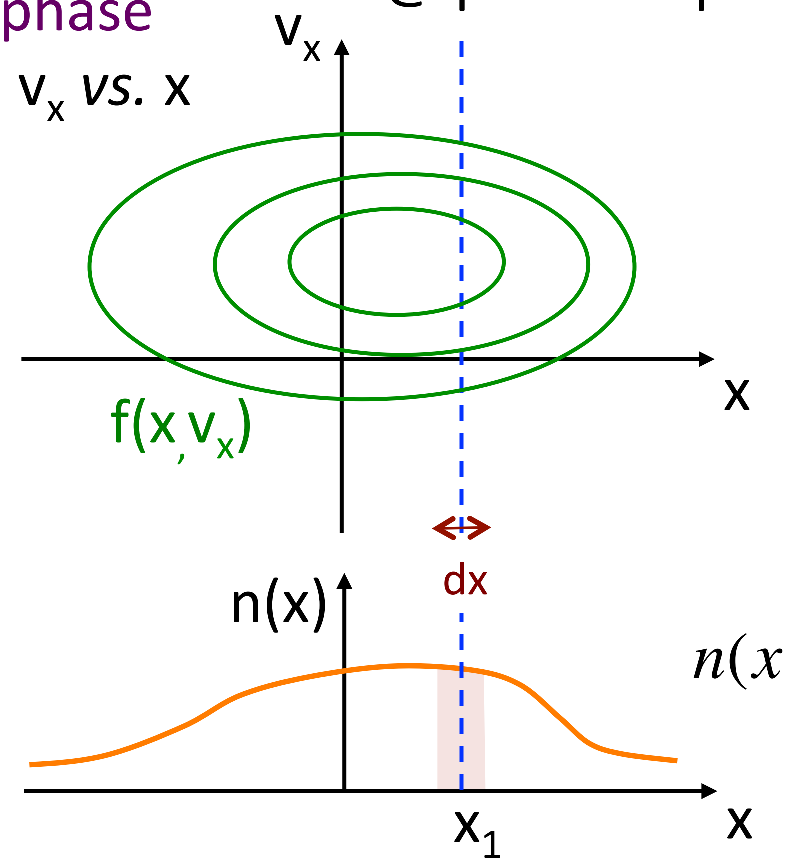
* non-relativistic: take c → ∞

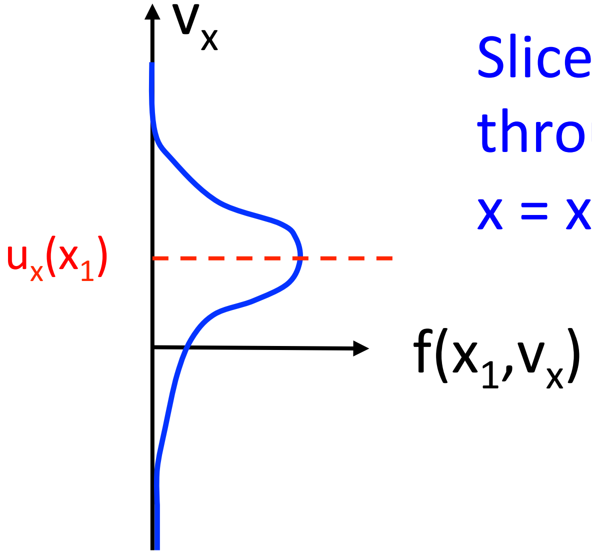
The distribution function

Fluid velocity =
mean velocity
@ point in space

$$u_x(x) = \frac{1}{n(x)} \int_{-\infty}^{\infty} \mathrm{v}_x f(x, \mathrm{v}_x)\, d\mathrm{v}_x$$

kinda-phase
space: v_x *vs.* x





Slice
through
$x = x_1$

$$n(x) = \int_{-\infty}^{\infty} f(x, \mathrm{v}_x)\, d\mathrm{v}_x$$

number density: integrate
over all* velocities

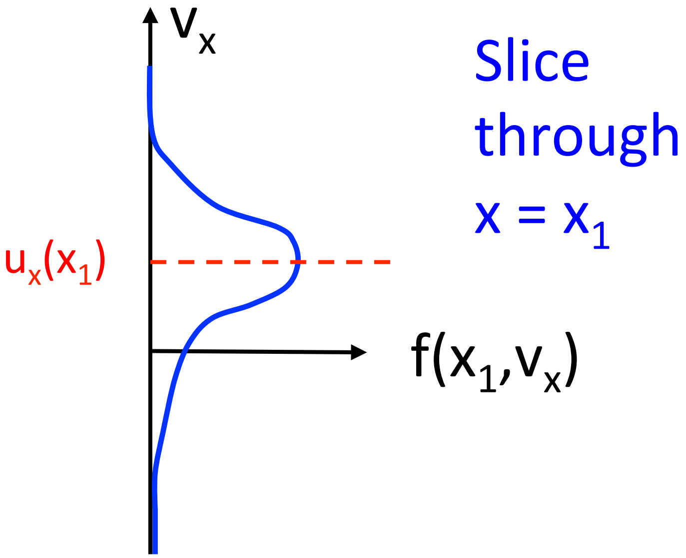
* non-relativistic: take c → ∞

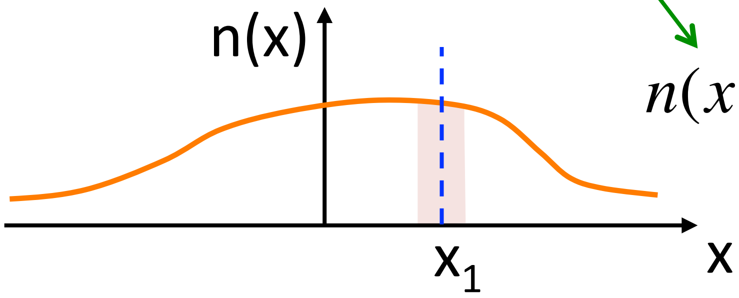
The distribution function

moments of f(v_x):

- 0^{th} : n
- 1^{st} : nu_x

$$n(x)u_x(x) = \int_{-\infty}^{\infty} \boxed{\mathrm{v}_x} f(x,\mathrm{v}_x)\,d\mathrm{v}_x$$





$$n(x) = \int_{-\infty}^{\infty} \boxed{1\cdot} f(x,\mathrm{v}_x)\,d\mathrm{v}_x$$

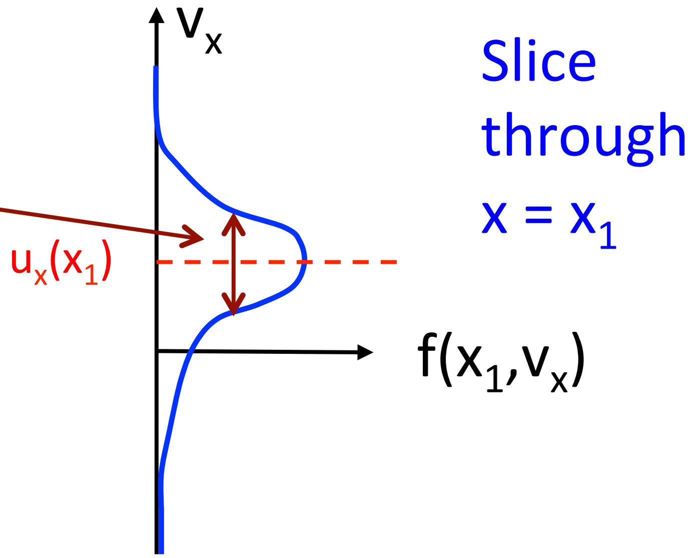
The distribution function

moments of $f(v_x)$:

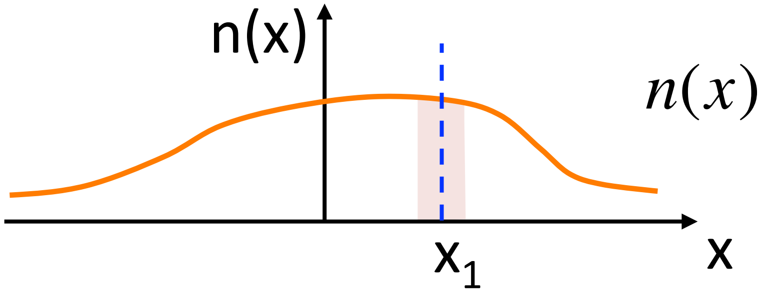
- 0^{th} : n
- 1^{st} : nu_x
- 2^{nd} : p/m

$$n(x)u_x(x) = \int_{-\infty}^{\infty} \mathrm{v}_x f(x,\mathrm{v}_x)\,d\mathrm{v}_x$$

$$p_{xx}(x) = m\int_{-\infty}^{\infty} (\mathrm{v}_x - u_x)^2 f(x,\mathrm{v}_x)\,d\mathrm{v}_x$$



Slice through $x = x_1$



$$n(x) = \int_{-\infty}^{\infty} 1\cdot f(x,\mathrm{v}_x)\,d\mathrm{v}_x$$

The 3d dist’n function

3 spatial dimensions: $\mathbf{x} = x_i = (x, y, z)$

3 velocity dimensions: $\mathbf{v} = v_i = (v_x, v_y, v_z)$

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6 Phase space dimensions

0th moment (scalar):
density

$$n(\mathbf{x}) = \int f(\mathbf{x}, \mathbf{v}) d^3\mathrm{v}$$

1st moment (vector):
fluid velocity

$$u_i(\mathbf{x}) = \frac{1}{n(\mathbf{x})} \int \mathrm{v}_i f(\mathbf{x}, \mathbf{v})\, d^3\mathrm{v}$$

2nd moment (tensor):
pressure

$$p_{ij}(\mathbf{x}) = m \int [\mathrm{v}_i - u_i(\mathbf{x})][\mathrm{v}_j - u_j(\mathbf{x})] f(\mathbf{x}, \mathbf{v})\, d^3\mathrm{v}$$

Pressure

$$\boxed{p_{ij}(\mathbf{x}) = m\int [\mathrm{v}_i - u_i(\mathbf{x})][\mathrm{v}_j - u_j(\mathbf{x})]\, f(\mathbf{x},\mathbf{v})\; d^3\mathrm{v}}$$

$$\tfrac{1}{2}\,\mathrm{Tr}(\overleftrightarrow{p}) = \tfrac{1}{2}\sum_i p_{ii} = \tfrac{1}{2} m \int |\mathbf{v}-\mathbf{u}|^2\, f(\mathbf{x},\mathbf{v})\; d^3\mathrm{v}$$

$$= \underbrace{\int \tfrac{1}{2} m |\mathbf{v}|^2\, f(\mathbf{x},\mathbf{v})\; d^3\mathrm{v}}_{\text{particle kinetic energy density } e} - \tfrac{1}{2} m |\mathbf{u}|^2 \underbrace{\int f(\mathbf{x},\mathbf{v})\; d^3\mathrm{v}}_{n(\mathbf{x})}$$

$$e = \overbrace{\int \tfrac{1}{2} m |\mathbf{v}|^2\, f(\mathbf{x},\mathbf{v})\; d^3\mathrm{v}} = \underbrace{\tfrac{1}{2} m\, n |\mathbf{u}|^2}_{\text{bulk kinetic energy}} + \underbrace{\tfrac{1}{2}\,\mathrm{Tr}(\overleftrightarrow{p})}_{\text{thermal energy } \varepsilon}$$

The pressure

$$\boxed{p_{ij}(\mathbf{x}) = m\int [\mathrm{v}_i - u_i(\mathbf{x})][\mathrm{v}_j - u_j(\mathbf{x})]\, f(\mathbf{x},\mathbf{v})\; d^3\mathrm{v}}$$

decompose

$$p_{ij}(\mathbf{x}) = \overbrace{\underbrace{p(\mathbf{x})}_{\text{scalar pressure}}\delta_{ij}}^{\text{isotropic}} - \overbrace{\underbrace{\sigma_{ij}(\mathbf{x})}_{\text{viscous stress tensor}}}^{\text{traceless anisotropic}}$$

force density

$$F_i = -\sum_j \frac{\partial p_{ij}}{\partial x_j} = \underbrace{-\frac{\partial p}{\partial x_i}}_{\text{pressure gradient: } -\nabla p} + \underbrace{\sum_j \frac{\partial \sigma_{ij}}{\partial x_j}}_{\text{viscous force}}$$

Pressure

$$\boxed{p_{ij}(\mathbf{x}) = m\int [\mathrm{v}_i - u_i(\mathbf{x})][\mathrm{v}_j - u_j(\mathbf{x})]\, f(\mathbf{x},\mathbf{v})\; d^3\mathrm{v}}$$

decompose
$$p_{ij}(\mathbf{x}) = \overbrace{\underbrace{p(\mathbf{x})}_{\text{scalar pressure}}\delta_{ij}}^{\text{isotropic}} - \overbrace{\underbrace{\sigma_{ij}(\mathbf{x})}_{\text{viscous stress tensor}}}^{\text{traceless anisotropic}}$$

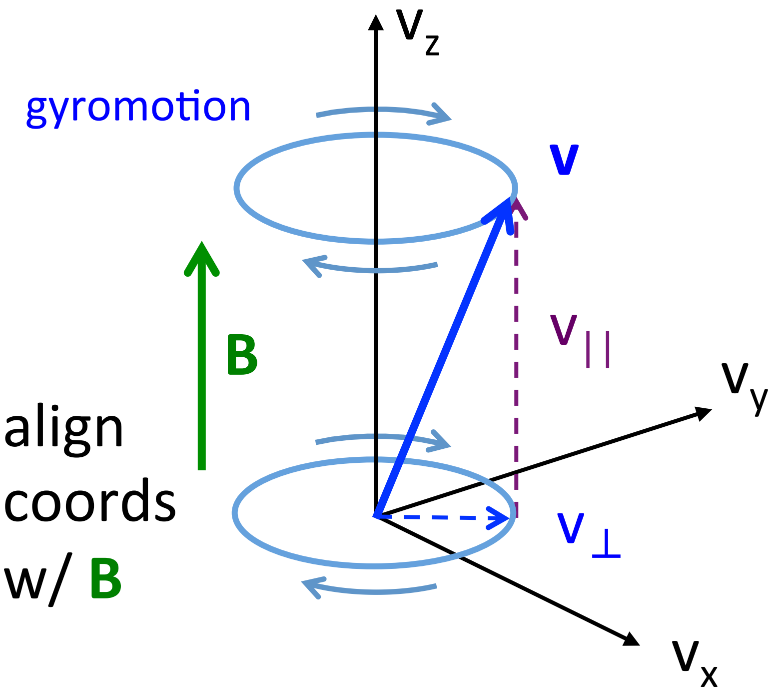
thermal energy:
$$\varepsilon = \tfrac{1}{2}\,\mathrm{Tr}(\overset{\leftrightarrow}{p}) = \tfrac{1}{2}\,p\,\underbrace{\mathrm{Tr}(\overset{\leftrightarrow}{I})}_{=3} - \cancel{\mathrm{Tr}(\overset{\leftrightarrow}{\sigma})}$$

➔ Ideal gas:
$$\boxed{\varepsilon = \tfrac{3}{2}\,p}$$

Plasma pressure

charged particle in frame co-moving w/ fluid (**u**=0)

$$\boxed{p_{ij} = m\int \mathrm{v}_i \mathrm{v}_j f(\mathbf{x},\mathbf{v})\, d^3\mathrm{v}}$$



align coords w/ **B**

gyromotion ➔

- $p_{xy} = p_{xz} = p_{yz} = 0$
- $p_{xx} = p_{yy} = p_\perp$
- $p_{zz} = p_{||}$

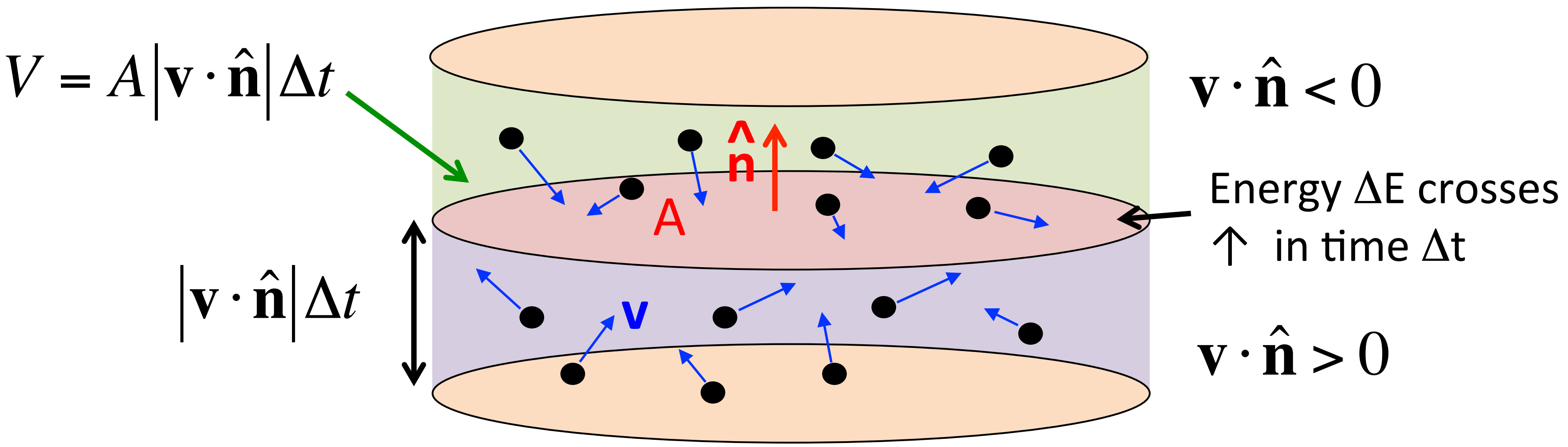
only 2 pressures

$$\overleftrightarrow{p} = \begin{bmatrix} p_\perp & 0 & 0 \\ 0 & p_\perp & 0 \\ 0 & 0 & p_\parallel \end{bmatrix}$$

$$p = \tfrac{1}{3}\mathrm{Tr}(\overleftrightarrow{p}) = \tfrac{2}{3}p_\perp + \tfrac{1}{3}p_\parallel$$

$$\overleftrightarrow{\sigma} = (p_\parallel - p_\perp)\left[\hat{\mathbf{b}}\hat{\mathbf{b}} - \tfrac{1}{3}\overleftrightarrow{I}\right]$$

Flux of particle energy



Energy ΔE crosses
↑ in time Δt

$$\Delta E = \overbrace{\int_{\mathbf{v}\cdot\hat{\mathbf{n}}>0} \tfrac{1}{2} m \mathrm{v}^2 A|\mathbf{v}\cdot\hat{\mathbf{n}}|\Delta t\, f(\mathbf{v})\, d^3\mathrm{v}}^{\text{energy in upward moving particles}} - \overbrace{\int_{\mathbf{v}\cdot\hat{\mathbf{n}}<0} \underbrace{\tfrac{1}{2} m \mathrm{v}^2}_{\text{energy}} \underbrace{A|\mathbf{v}\cdot\hat{\mathbf{n}}|\Delta t}_{\text{volume}}\, f(\mathbf{v})\, d^3\mathrm{v}}^{\text{energy in downward moving particles}}$$

$$= A\Delta t\, \hat{\mathbf{n}}\cdot\int \tfrac{1}{2} m \mathrm{v}^2 \mathbf{v}\, f(\mathbf{v})\, d^3\mathrm{v} = A\Delta t\, \hat{\mathbf{n}}\cdot\vec{\Gamma}_E$$

particle energy flux

[erg s^{-1} cm^{-2}]

$$\boxed{\vec{\Gamma}_E = \int \tfrac{1}{2} m \mathrm{v}^2 \mathbf{v}\, f(\mathbf{v})\, d^3\mathrm{v}}$$

Particle energy flux

$$\boxed{\vec{\Gamma}_E = \int \tfrac{1}{2} m \mathrm{v}^2 \mathbf{v}\, f(\mathbf{v})\, d^3\mathrm{v}}$$

introduce $\mathbf{v} = (\mathbf{v}-\mathbf{u}) + \mathbf{u}$

NB: $\int \big(\mathbf{v}-\mathbf{u}\big)\, f(\mathbf{v})\, d^3\mathrm{v} = 0$

1st moment of f(**v**) vanishes

use 2nd moment $p_{ij} = m\int (\mathrm{v}_i - u_i)(\mathrm{v}_j - u_j)\, f(\mathbf{v})\, d^3\mathrm{v} = p\,\delta_{ij} - \sigma_{ij}$

0th moment

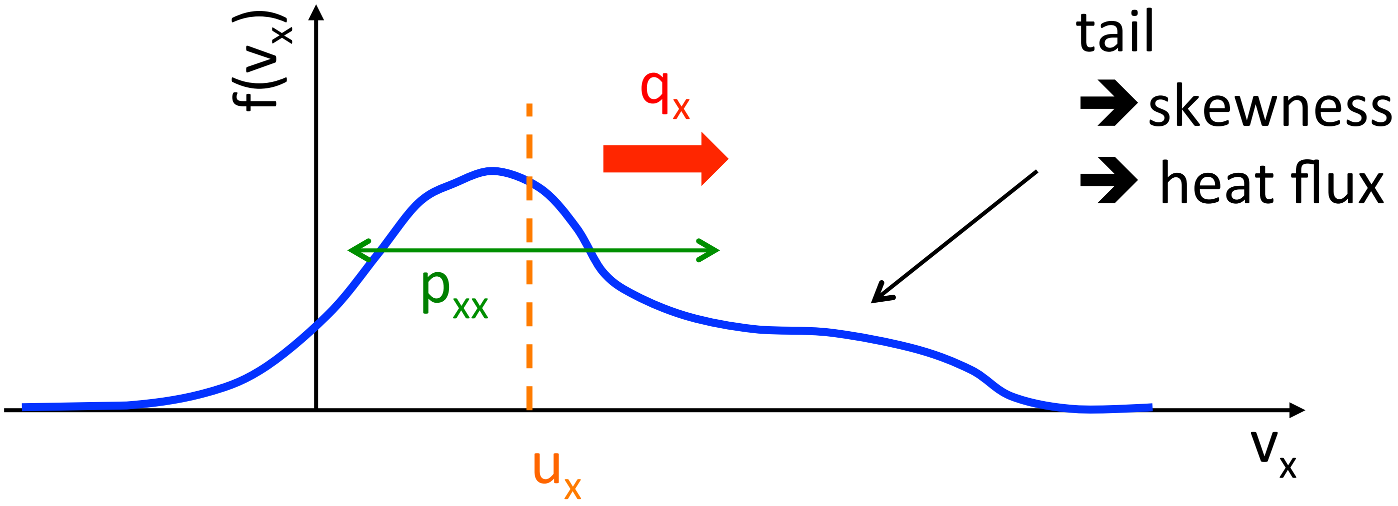
2nd moments

$$\vec{\Gamma}_E = \underbrace{\tfrac{1}{2} mn\,\mathbf{u}|\mathbf{u}|^2}_{\text{bulk kinetic energy flux}} + \underbrace{\tfrac{5}{2} p\mathbf{u}}_{\text{enthalpy flux}} - \underbrace{\overleftrightarrow{\sigma}\cdot\mathbf{u}}_{\text{viscous work}} + \underbrace{\tfrac{1}{2} m\int (\mathbf{v}-\mathbf{u})|\mathbf{v}-\mathbf{u}|^2 f(\mathbf{v})\, d^3\mathrm{v}}_{\text{heat flux} = \mathbf{q}}$$

= (ε + p)**u**

3rd moment of f(**v**)
– a.k.a. **skewness**

Moments of f(**v**)



mom.	probability	fluid	
0th	integral	density	n
1st	mean	fluid velocity	u_x
2nd	variance	pressure	p_{xx}
3rd	skewness	heat flux	q_x
4th	Kurtosis	? (no name)	

What is a fluid?

Described by moments of **f(x,v)**

n(**x**), **u**(**x**), & p(**x**)

which evolve according to fluid equations

$$\text{Exactly correct}\ \left\{\begin{aligned} &\frac{\partial n}{\partial t}+\nabla\cdot(n\mathbf{u})=0 \\ &mn\frac{\partial \mathbf{u}}{\partial t}+mn(\mathbf{u}\cdot\nabla)\mathbf{u}=-\nabla p+\boxed{\nabla\cdot\overleftrightarrow{\sigma}} \\ &\frac{\partial p}{\partial t}+\mathbf{u}\cdot\nabla p=-\tfrac{5}{3}p\nabla\cdot\mathbf{u}+\boxed{\tfrac{2}{3}\nabla\mathbf{u}:\overleftrightarrow{\sigma}}-\tfrac{2}{3}\boxed{\nabla\cdot\mathbf{q}} \end{aligned}\right.$$

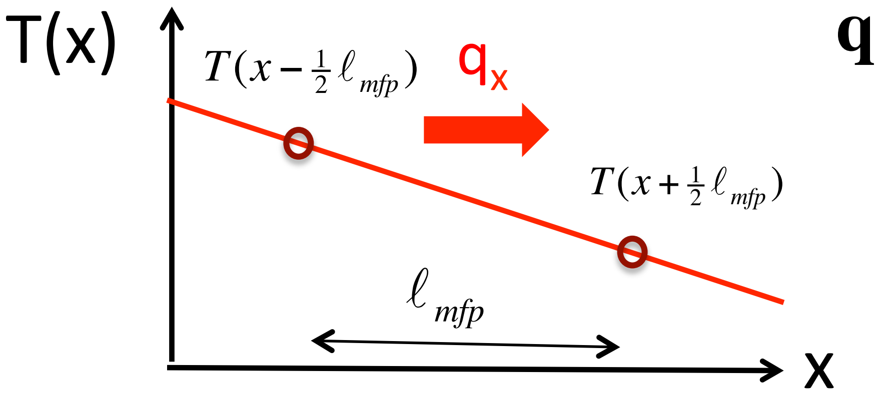
– but not “closed”: evolution depends on “higher moments” σ_{ij} & q_i

How can we find these?

4 ways to close the fluid eqs.

Strategy 1: relate σ_{ij} & q_i to spatial derivatives of n(**x**), u(**x**), & p(**x**)

Works when particle mfp is small compared to gradient length scales – f(**x,v**) will be close to Maxwellian given by n(**x**), u(**x**), & p(**x**). **Small** departures produce σ_{ij} & q_i



$$\mathbf{q} = -\kappa \nabla T = -\frac{\kappa}{k_b} \nabla \left(\frac{p}{n} \right)$$

thermal conductivity $\kappa = \frac{3}{2} \ell_{mfp} n k_b \sqrt{\frac{k_b T}{m}}$

Similar approach for σ_{ij} – coefficient = viscosity

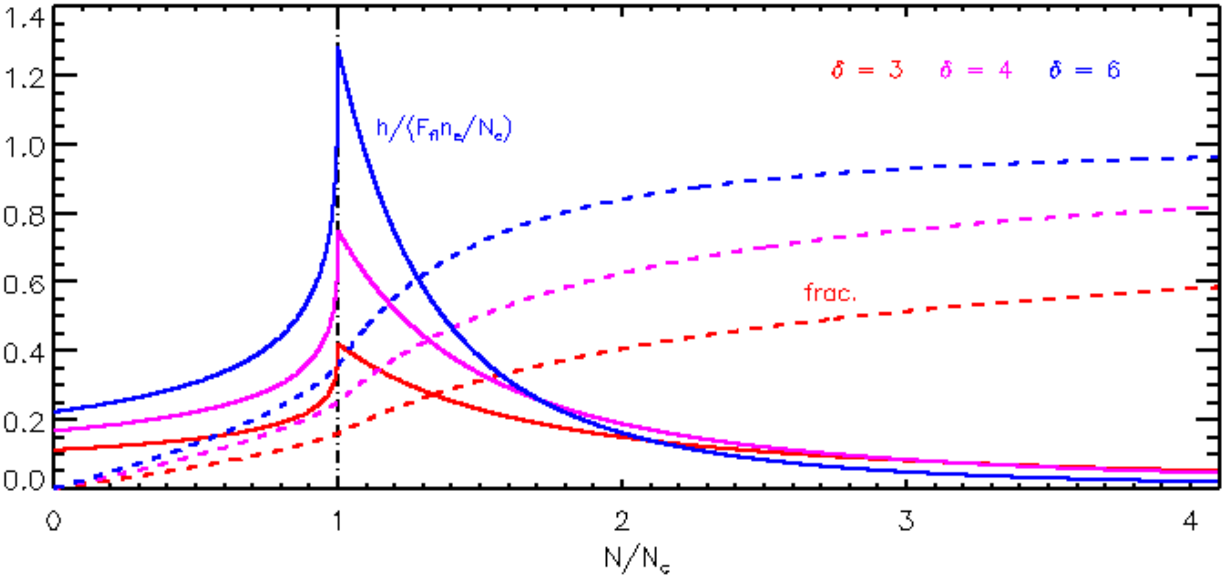
4 ways to close the fluid eqs.

Strategy 2: take $\sigma_{ij}(\mathbf{x})$ & $q_i(\mathbf{x})$ as given – use in fluid eqs.

$$h = -\nabla\cdot\mathbf{q} = \frac{\delta-2}{6}\left(\frac{n_e F_{\rm fl}}{\mu_0 N_c}\right)\left(\frac{N}{\mu_0 N_c}\right)^{-\delta/2}\begin{cases} B\left(\frac{N}{\mu_0 N_c};\frac{\delta}{2},\frac{1}{3}\right) & , \quad N<\mu_0 N_c \\ B\left(\frac{\delta}{2},\frac{1}{3}\right)=\frac{\Gamma(\frac{1}{2}\delta+\frac{1}{3})}{\Gamma(\frac{1}{2}\delta)\Gamma(\frac{1}{3})} & , \quad N>\mu_0 N_c \end{cases}$$

➔ fluid eqs. do include non-thermal electrons via $q_i(\mathbf{x})$

$q_i(\mathbf{x})$ specified via parameters δ, μ_0, E_c, & $F_{\rm fl}$



4 ways to close the fluid eqs.

Strategy 3: new eqs. for evolution of $\sigma_{ij}(\mathbf{x})$ & $q_i(\mathbf{x})$

- CGL eqns. → separate "energy eqs." for $p_{||}$ & $p_\perp$ ➔ $\sigma_{ij}(\mathbf{x})$

$$p = \tfrac{1}{3}\mathrm{Tr}(\overleftrightarrow{p}) = \tfrac{2}{3}p_\perp + \tfrac{1}{3}p_{||} \qquad \overleftrightarrow{\sigma} = (p_{||} - p_\perp)\left[\hat{\mathbf{b}}\hat{\mathbf{b}} - \tfrac{1}{3}\overleftrightarrow{I}\right]$$

- **But** eqn. for 3rd moment, $q_i(\mathbf{x})$, involves 4th moment…
- … eqn. for 4th moment involves 5th moment …
- etc. – ∞ hierarchy = no closure
- ∞ information contained in f(**v**) can only be captured by ∞ moments

4 ways to close the fluid eqs.

Strategy 4: follow evolution of f(**x**,**v**)

Fokker-Planck equation

(next 2 lectures)

$$\frac{\partial f}{\partial t} = \mathrm{stuff}$$

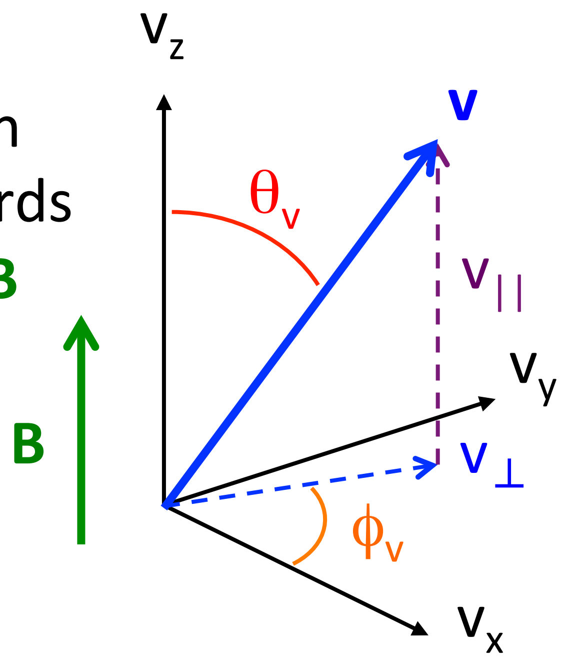
PROBLEMS:

- PDE in 6d [sic] phase space – 6+1d PDE
  - gyromotion reduces to 5+1d
  - single field line: 3+1d – still big!
- Can include very short time scales:

$$\tau^{-1} \sim \nu_{col}, \ \Omega_c, \ \omega_p$$

velocity space in polar coords

align coords w/ **B**



θ_v: pitch angle $\in$ (0, π)

$\mu = \cos(\theta_v)$: pitch angle cosine $\in (-1, 1)$

velocity space volume elements:

$$\left\{ \begin{aligned} d^3v &= dv_x dv_y dv_z \\ &= v^2\, dv\, d\Omega_v \\ &= v^2\, dv \sin(\theta_v)\, d\theta_v\, d\phi_v \\ &= v^2\, dv\, d\mu\, d\phi_v \end{aligned} \right.$$

integrate over gyrophase:

$$\mathrm{v}^2 d\mathrm{v}\, d\mu \int_0^{2\pi} f(\mathbf{v})\, d\phi_\mathrm{v} = f(\mathrm{v},\mu)\, d\mathrm{v}\, d\mu$$

gyromotion ➔ no dep'nce on ϕ_v

$$\boxed{f(\mathrm{v},\mu) = 2\pi \mathrm{v}^2 f(\mathbf{v})}$$

integrate over pitch angles:

$$\mathrm{v}^2 d\mathrm{v} \int f(\mathbf{v})\, d\mu\, d\phi_\mathrm{v} = f(\mathrm{v})\, d\mathrm{v}$$

$$\boxed{f(\mathrm{v}) = 4\pi \mathrm{v}^2 \left\langle f(\mathbf{v}) \right\rangle_{\Omega_v}}$$

Different distributions you meet

density of particles

$$n = \int f(\mathbf{v})\,d^3\mathrm{v} \; = \int f(\mathrm{v},\mu)\,d\mathrm{v}\,d\mu \; = \int_0^\infty f(\mathrm{v})\,d\mathrm{v} = \int_0^\infty f(E)\,dE$$

Energy distribution function

$$f(E) = f(\mathrm{v})\frac{d\mathrm{v}}{\mathrm{d}E} = \frac{f(\mathrm{v})}{m\mathrm{v}} = \frac{4\pi\mathrm{v}}{m}\left\langle f(\mathbf{v})\right\rangle_{\Omega_v}$$

Energy density

$$e = \int \tfrac{1}{2}m\left|\mathbf{v}\right|^2 f(\mathbf{v})\,d^3\mathrm{v} \; = \; \tfrac{1}{2}m\int_0^\infty \mathrm{v}^2 f(\mathrm{v})\,d\mathrm{v} = \int_0^\infty E\; f(E)\,dE$$

Different distributions you meet

energy flux along magnetic field – $\Gamma_{E,z}$

$$\Gamma_{E,z} = \tfrac{1}{2} m \int \mathrm{v}_z |\mathbf{v}|^2 f(\mathbf{v})\, d^3\mathrm{v} = \tfrac{1}{2} m \int \mu \mathrm{v}^3 f(\mathrm{v},\mu)\, d\mathrm{v}\, d\mu$$

$$= \int E\, F(E,\mu)\, dE\, d\mu$$

flux spectrum: $F(E,\mu) = \mu \mathrm{v}\, f(\mathrm{v},\mu) \dfrac{d\mathrm{v}}{\mathrm{d}E} = \dfrac{\mu}{m} f(\mathrm{v},\mu)$

= flux of electrons per E per μ [e^- /erg/s/cm^2]

- most directly probed by flare observations
- typical model: $F(E,\mu) \sim E^{-\delta}$
- $\delta > 2$ in order that $\Gamma_{E,z} < \infty$

Entropy

Entropy per unit volume:

$$\boxed{s(\mathbf{x}) = -k_b \int \ln\left[f(\mathbf{x},\mathbf{v})\right] f(\mathbf{x},\mathbf{v})\, d^3\mathrm{v}}$$

2nd Law of Thermo: short-range interactions between particles (i.e. collisions) can only **increase** *s* at a point

- **elastic collisions** must do so while conserving mass (mn), momentum ($mn\mathbf{u}$), and energy (e).
- after sufficiently many collisions, *s* will reach a **maximum** – max. subject to constraints on n, $\mathbf{u}$ & e
- ➔ collisions drive $f(\mathbf{x},\mathbf{v})$ to steady state defined by maximum *s* – **local thermodynamic equilibrium***

* Stricter usage demands particles of all species, **and radiation** be in equilibrium with one another

Entropy density

$$\boxed{s(\mathbf{x}) = -k_b \int \ln\left[f(\mathbf{x},\mathbf{v})\right] f(\mathbf{x},\mathbf{v})\, d^3\mathrm{v}}$$

maximize entropy subject to conservation:

number: $\boxed{\alpha}\,\delta n = \int \boxed{\alpha}\,\boxed{\delta f}\, d^3\mathrm{v} = 0$

momentum: $\sum_i \boxed{\mu_i}\,\delta\left(mnu_i\right) = \int m\,\boxed{\vec{u}}\cdot \mathbf{v}\,\boxed{\delta f}\, d^3\mathrm{v} = 0$

energy: $\boxed{\beta}\,\delta e = \int \boxed{\beta}\,\tfrac{1}{2} m\left|\mathbf{v}\right|^2 \boxed{\delta f}\, d^3\mathrm{v} = 0$

Lagrange multipliers

variation of f(**x**,**v**)

LTE: maximize

$$\boxed{s(\mathbf{x}) = \text{-}k_b \int \ln\left[f(\mathbf{x},\mathbf{v})\right] f(\mathbf{x},\mathbf{v})\, d^3\mathrm{v}}$$

subject to
constraints:

number: $$\alpha\,\delta n = \int \alpha\, \delta f\ d^3\mathrm{v} = 0$$

momentum: $$\sum_i \mu_i\, \delta\left(m n u_i\right) = \int m\, \vec{\mu}\cdot\mathbf{v}\ \delta f\ d^3\mathrm{v} = 0$$

energy: $$\beta\,\delta e = \int \beta\, \tfrac{1}{2} m \left|\mathbf{v}\right|^2\ \delta f\ d^3\mathrm{v} = 0$$

Max: $$\boxed{\delta s = -k_b \int \left[\ln f + (1+\alpha) + m\vec{\mu}\cdot\mathbf{v} + \beta\,\tfrac{1}{2} m\left|\mathbf{v}\right|^2\right] \delta f\ d^3\mathrm{v} = 0}$$

→ $$f \propto \exp\left[-m\vec{\mu}\cdot\mathbf{v} - \beta\,\tfrac{1}{2} m\left|\mathbf{v}\right|^2\right]$$

The “Max” in Maxwellian
is for **entropy**

define: $$\left\{\begin{array}{l} \vec{\mu} = \beta\,\mathbf{u} \\ \beta = \dfrac{1}{k_b T} \end{array}\right.$$

→ $$\boxed{f(\mathbf{v}) = \frac{n}{(2\pi k_b T/m)^{3/2}} \exp\left[-\frac{\tfrac{1}{2} m \left|\mathbf{v}-\mathbf{u}\right|^2}{k_b T}\right]}$$

The Maxwellian

$$\boxed{f(\mathbf{x},\mathbf{v}) = \frac{n(\mathbf{x})}{(2\pi k_b T / m)^{3/2}} \exp\left[-\frac{\frac{1}{2} m \left| \mathbf{v} - \mathbf{u}(\mathbf{x}) \right|^2}{k_b T(\mathbf{x})} \right]}$$

$$p_{ij} = m \int (\mathrm{v}_i - u_i)(\mathrm{v}_j - u_j) \, f(\mathbf{x},\mathbf{v}) \; d^3\mathrm{v} = n k_b T \delta_{ij}$$

$$q_i = \tfrac{1}{2} m \int (\mathrm{v}_i - u_i) \left| \mathbf{v} - \mathbf{u} \right|^2 f(\mathbf{x},\mathbf{v}) \; d^3\mathrm{v} = 0$$

➔ideal fluid: inviscid: $\overleftrightarrow{\sigma} = 0$

no heat flux: $\vec{q} = 0$

closed fluid equations

isotropic pressure $p = n k_b T$

The Maxwellian

in fluid ref.
frame (**u**=0)

$$f(\mathbf{v}) = \frac{n}{(2\pi k_b T / m)^{3/2}} \exp\left[-\frac{\frac{1}{2} m |\mathbf{v}|^2}{k_b T} \right]$$

$$f(\mathrm{v}) = 4\pi \mathrm{v}^2 \left\langle f(\mathbf{v}) \right\rangle_{\Omega_v} = \frac{4\pi n}{(2\pi k_b T / m)^{3/2}} \mathrm{v}^2 \exp\left[-\frac{m \mathrm{v}^2}{2 k_b T} \right]$$

$$f(E) = \frac{4\pi \mathrm{v}}{m} \left\langle f(\mathbf{v}) \right\rangle_{\Omega_v} = \frac{2}{\sqrt{\pi}} \frac{n}{(k_b T)^{3/2}} \sqrt{E} \exp\left[-\frac{E}{k_b T} \right]$$

checks:

$$\int_0^\infty f(E)\, dE = \frac{2}{\sqrt{\pi}} n \boxed{\int_0^\infty s^{1/2} e^{-s}\, ds} = n$$

$$\Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2} \Gamma\left(\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\int_0^\infty E\, f(E)\, dE = \frac{2}{\sqrt{\pi}} n k_b T \boxed{\int_0^\infty s^{3/2} e^{-s}\, ds} = \tfrac{3}{2} n k_b T$$

$$\Gamma\left(\tfrac{5}{2}\right) = \tfrac{3}{2} \Gamma\left(\tfrac{3}{2}\right) = \frac{3\sqrt{\pi}}{4}$$

The Maxwellian

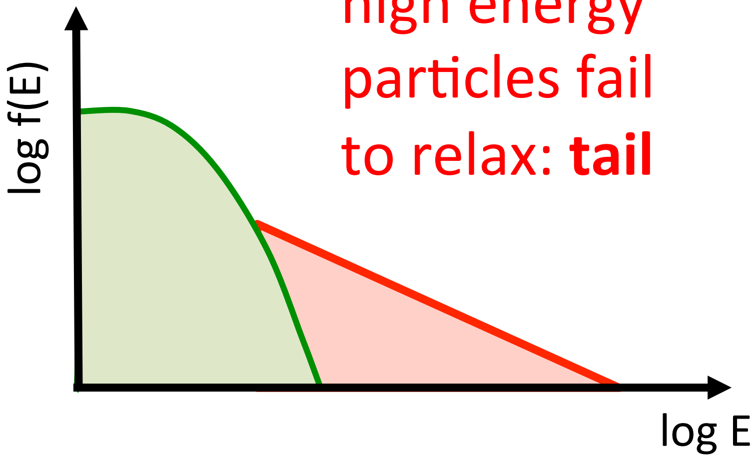
$$\boxed{f(E)=\frac{2}{\sqrt{\pi}}\frac{n}{(k_bT)^{3/2}}\sqrt{E}\exp\left[-\frac{E}{k_bT}\right]}$$

Occurs when **something** maximizes entropy while conserving, mass, momentum and energy – i.e. elastic collisions between particles (see 2nd Law Thermo) – collisions **“relax”** distribution toward a Maxwellian

collision rate scales inversely w/ E

$$\nu_{\rm col}=n\sigma \mathrm{v}=\frac{2\pi e^4 n\Lambda}{m^{1/2}}\frac{1}{E^{3/2}}$$

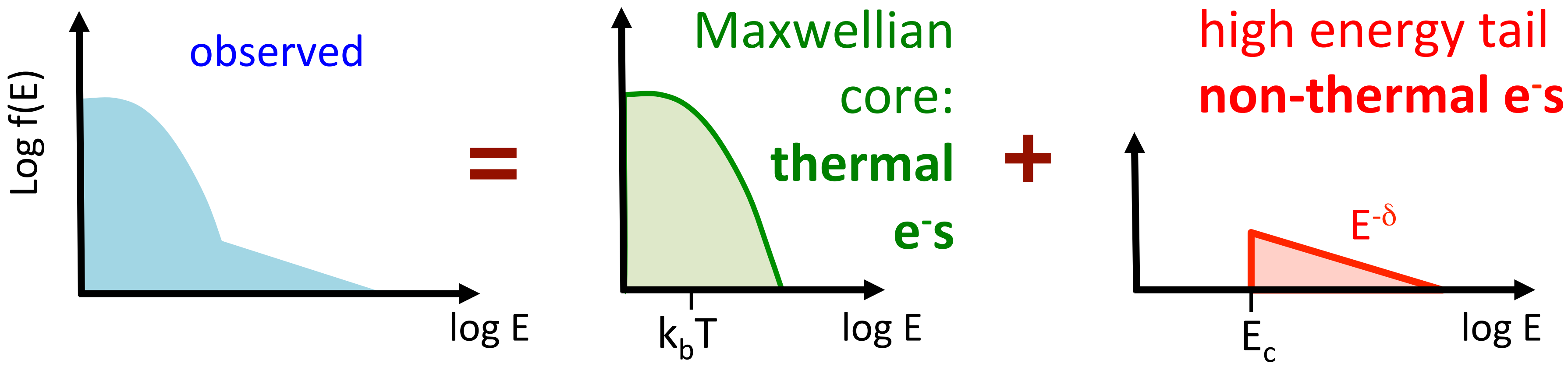
low energy particles relax quickly: **Maxwellian core**



high energy particles fail to relax: **tail**

An artificial decomposition …

… but common, useful, enlightening, …

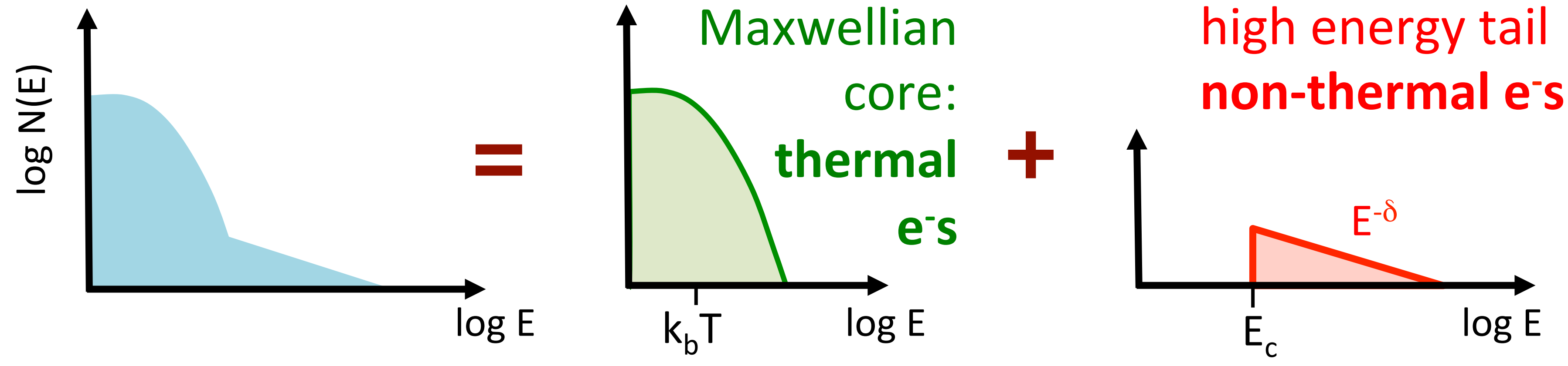


$$\overbrace{f(E)} = \overbrace{\frac{2}{\sqrt{\pi}}\frac{n_{th}}{(k_bT)^{3/2}}\sqrt{E}\exp\left[-\frac{E}{k_bT}\right]} + \overbrace{f_{nt}(E)}$$

density of NT e$^-$s: $\int_0^\infty f_{nt}(E)\,dE$ = n_{nt} << n_{th} typically

An artificial decomposition …

… but common, useful, enlightening, …



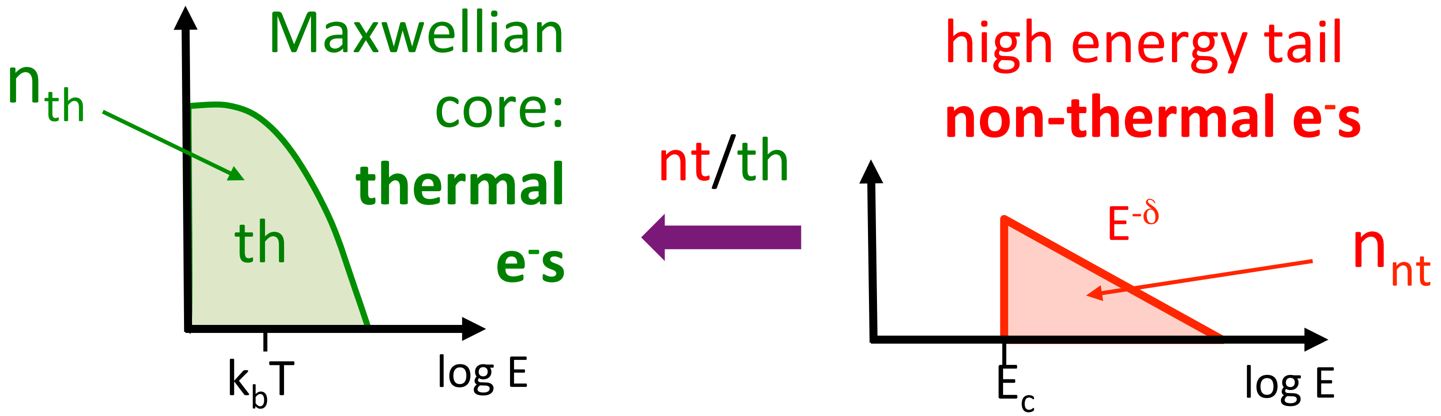
$$\overbrace{f(E)}^{} = \overbrace{\frac{2}{\sqrt{\pi}}\frac{n_{th}}{(k_bT)^{3/2}}\sqrt{E}\exp\left[-\frac{E}{k_bT}\right]}^{} + \overbrace{f_{nt}(E)}^{}$$

NT energy density:

$$\varepsilon_{nt} = \int_0^\infty E\, f_{nt}(E)\, dE = \frac{\delta-1}{\delta-2}E_c\, n_{nt}$$

$$\frac{\varepsilon_{nt}}{\varepsilon_{th}} \approx \overset{<<1}{\boxed{\frac{n_{nt}}{n_{th}}}}\ \overset{>>1}{\boxed{\frac{E_c}{k_bT}}}$$

Collisions



Maxwellian core: **thermal e^-s**

high energy tail **non-thermal e^-s**

th/th: **no effect** – Maxwellian is steady state (attractor) of collisions (cannot further increase *s*)

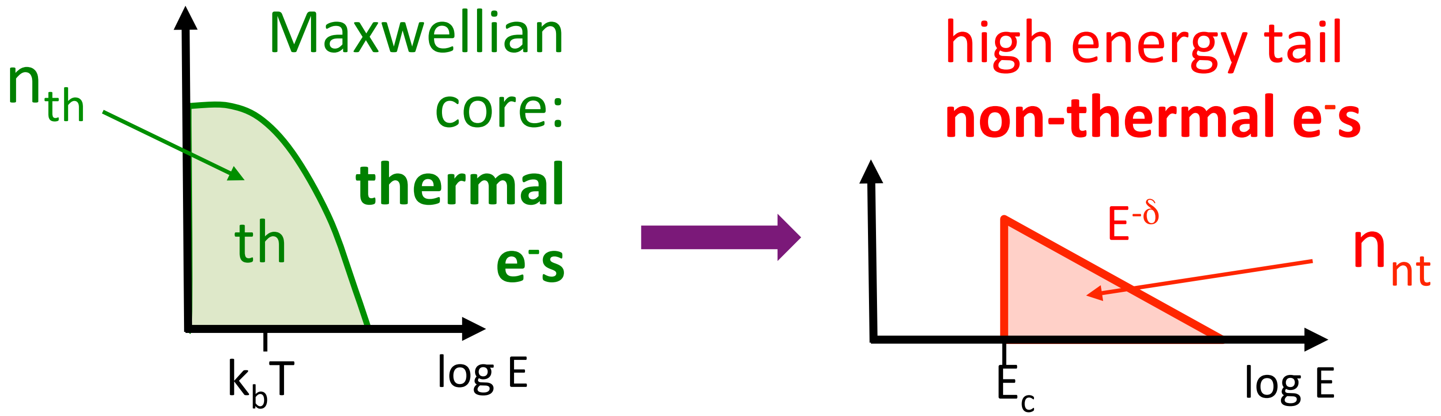
nt/nt: negligible vs. nt/th – $n_{nt} << n_{th}$

nt/th: transfers Δn $n_{nt} \rightarrow n_{th}$

transfers $\Delta\varepsilon$ $\varepsilon_{nt} \rightarrow \varepsilon_{th}$ – thermalization

see NT e^- heating h(s) in lecture 11

Acceleration



pre-flare state:

- thermal plasma: $n_{nt} = 0$
- quiescent AR: $n_{th} \sim 3 \times 10^9$ cm^{-3} , $T \sim 3 \times 10^6$ K

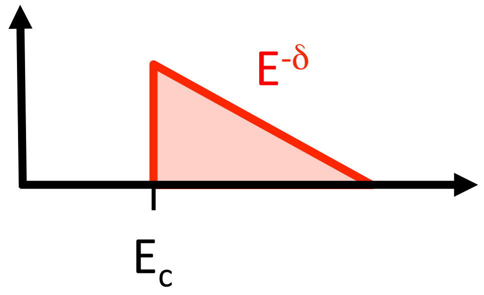
during flare: something

- transfers Δn $n_{th} \rightarrow n_{nt}$
- adds $\Delta\varepsilon$ to ε_{nt}
- sets NT parameters δ & E_c

Qs:

- **What something?**
- Whence $\Delta\varepsilon$?
- Why is $f_{nt}(E)$ a power-law?
- What sets n_{nt}, δ, E_c ?

Acceleration



- **What something?**
- Whence $\Delta\varepsilon$?
- Why is $f_{nt}(E)$ a power-law?
- What sets n_{nt}, δ, E_c ?

Starting points for As:

- Shocks (Fermi), DC **E** field, wave-particle interactions, …
- Magnetic reconnection ➔ magnetic energy, bulk KE, …
- ??
- ??

These answers must lie in the evolution of f(E) or f(**x,v**) i.e. Fokker-Planck

Summary

- Collection of particles described by distribution function f(**x**,**v**)
- Moments of f(**x**,**v**) yield fluid properties
- Fluid equations capture most of behavior – heat flux **q** is notable exception (sometimes)
- Collisions drive f(**x**,**v**) toward Maxwellian – does so slowly for high-energy particles: the non-thermal tail

Next: How tail of f(**x**,**v**) evolves in time & forms a power-law: The Fokker-Planck equation