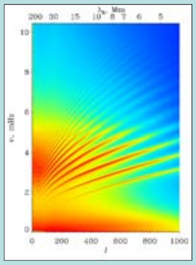
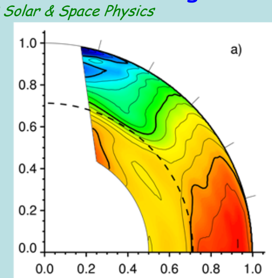
ASTR 7500: Solar & Stellar Magnetism

Hale CGEP Solar & Space Physics





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Lecture 22 Tues 11 Apr 2013

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Lecture 22
Acoustic-Gravity Waves

- Restoring Forces
  - Which ones are important?
  - Pressure fluctuations and sound waves
  - Buoyancy fluctuations and internal gravity waves
  - Surface gravity waves
- Wave Equation for Acoustic-Gravity Waves
  - Linearize about a steady background atmosphere
  - Derive a buoyancy force equation
  - Derive a wave equation for the dilation
  - Examine the special case of the f mode
- Local Dispersion Relation
  - Standard form
  - Sphericity

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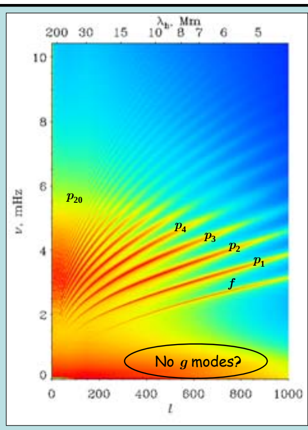

Solar Acoustic Spectrum

f – Fundamental mode (surface gravity wave)

p_n – acoustic mode (Pressure wave)

g_n – gravity mode (internal Gravity wave)

Reminder

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Part 1

Restoring Forces

Momentum Equation

Consider the inviscid form of the Navier-Stokes equation, in the presence of gravity in a rotating reference frame.

$$\rho\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)\vec{v} = -\vec{\nabla}P + \rho\vec{g} - 2\rho\,\vec{\Omega}\times\vec{v} + \frac{1}{4\pi}(\vec{\nabla}\times\vec{B})\times\vec{B}$$

Pressure perturbations drive sound waves

Buoyancy perturbations drive gravity waves

The Coriolis force drives Rossby waves

The Lorentz force drives Alfvén waves

$$\vec{g} \equiv \vec{g}_{\rm N} + \Omega^2\vec{R}$$

Newtonian gravity and the centrifugal force can be combined into an effective gravity

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Which Forces are Important

In the solar interior, when compared to a 5 minute period oscillation, the time scale associated with the Lorentz force and Coriolis force are small, and those forces can be ignored.

Acoustic Time Scale $(kc)^{-1}$

$$\frac{\text{acoustic time scale}}{\text{wave period}} = \frac{\omega}{kc} \sim 1$$

Buoyancy Time Scale N^{-1}

$$\frac{\text{acoustic time scale}}{\text{buoyancy time scale}} = \frac{N}{\omega} \sim \frac{0.3\ \text{mHz}}{3\ \text{mHz}} = 10^{-1}$$

Alfvén Time Scale $(kV_A)^{-1}$

$$\frac{\text{acoustic time scale}}{\text{Alfvén time scale}} = \frac{V_A}{c} \sim \frac{20\ \text{m/s}}{200\ \text{km/s}} = 10^{-4}$$

Rotational Period Ω^{-1}

$$\frac{\text{acoustic time scale}}{\text{rotational period}} = \frac{\Omega}{\omega} \sim \frac{5\,\text{min}}{1\ \text{month}} = 10^{-4}$$

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Some Preliminaries

We will look at pressure and buoyancy fluctuations separately and briefly investigate the properties of the waves that each enables. But, first we need to discuss several preliminary issues.

Lagrangian and Eulerian Derivatives
Consider some property f of the fluid. f could be scalar such as the mass density or pressure; but, it could also be a vector such as the velocity.

The *Eulerian* time derivative of f is the rate of change one sees at a fixed position,

Eulerian time derivative $\dfrac{\partial f}{\partial t}$

The Lagrangian time derivative (or the advective derivative) is the rate of change one sees when following a parcel of fluid

Lagrangian time derivative $\dfrac{Df}{Dt} = \dfrac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f$

Note: If the property f is homogeneous, then the two derivatives are the same,

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t}$$

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Adiabaticity

Adiabatic Oscillations
We will assume that the oscillation occurs rapidly enough that we can ignore energy loss during wave cycle. This means that the thermal loss time is much longer than a wave period.

$$\tau_{\text{thermal}} \gg \omega^{-1}$$

The thermal loss time is determined by thermal conduction, radiative losses, ionization, viscosity, etc.

Such oscillations are called *adiabatic*, and we know from thermodynamics that the pressure and density are related

Adiabatic Energy Equation $\rho^{-\gamma} P = \text{constant}$

Adiabatic Exponent $\gamma = 5/3$ (for an ideal gas)

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Energy Equation

$$\rho^{-\gamma} P = \text{constant}$$

The energy equation can be written in a more useful form by taking the time derivative for a moving fluid parcel,

$$\frac{D}{Dt}\left(\rho^{-\gamma} P\right) = 0$$

product & chain rule

$$\rho^{-\gamma}\left(\frac{DP}{Dt} - \frac{\gamma P}{\rho}\frac{D\rho}{Dt}\right) = 0$$

Sound Speed

$$c^2 \equiv \frac{\gamma P}{\rho} = \gamma R_{\text{gas}} T$$

This last equation is equivalent to . . . $\dfrac{DP}{Dt} = c^2 \dfrac{D\rho}{Dt}$

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Pressure Fluctuations and Sound Waves

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Pressure Fluctuations

Sound waves are an interplay between inertia and pressure fluctuations. In the absence of gravity, adiabatic sound waves obey:

Momentum Equation $\rho \dfrac{D\vec{v}}{Dt} = -\vec{\nabla} P$

Continuity Equation $\dfrac{D\rho}{Dt} = -\rho \vec{\nabla} \cdot \vec{v}$

(Adiabatic) Energy Equation $\dfrac{DP}{Dt} = c^2 \dfrac{D\rho}{Dt}$

The equations of continuity and adiabaticity can be combined to give an alternate form of the energy equation $\dfrac{DP}{Dt} = -\rho c^2 \vec{\nabla} \cdot \vec{v}$

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Linearize about a Homogeneous Background

Assume that the oscillations are small fluctuations around a static, steady, homogeneous atmosphere.

$$P(\vec{x},t) = P_0 + P_1(\vec{x},t)$$

$$\rho(\vec{x},t) = \rho_0 + \rho_1(\vec{x},t)$$

$$\vec{v}(\vec{x},t) = \vec{v}_1(\vec{x},t)$$

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Linearized Equations

Insert the previous expansions into the fluid equations and discard nonlinear terms

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}$$

Momentum Equation $\quad \rho \frac{D\vec{v}}{Dt} = -\vec{\nabla} P \quad \rightarrow \quad \rho_0 \frac{\partial \vec{v}_1}{\partial t} = -\vec{\nabla} P_1$

Continuity Equation $\quad \frac{D\rho}{Dt} = -\rho \vec{\nabla}\cdot\vec{v} \quad \rightarrow \quad \frac{\partial \rho_1}{\partial t} = -\rho_0 \vec{\nabla}\cdot\vec{v}_1$

(Adiabatic) Energy Equation $\quad \frac{DP}{Dt} = -\rho c^2 \vec{\nabla}\cdot\vec{v} \quad \rightarrow \quad \frac{\partial P_1}{\partial t} = -\rho_0 c^2 \vec{\nabla}\cdot\vec{v}_1$

Linear Homogenous

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Wave Equation

We can combine the momentum and energy equations to obtain a single PDEs for the velocity

$$\left.\begin{aligned} \rho_0 \frac{\partial \vec{v}_1}{\partial t} &= -\vec{\nabla} P_1 \\ \frac{\partial P_1}{\partial t} &= -\rho_0 c^2 \vec{\nabla}\cdot\vec{v}_1 \end{aligned}\right\} \quad \left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2\right)\vec{v}_1 = 0$$

Inertial Term

Pressure Gradient Force

Remember where the terms came from

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Acoustic Dispersion Relation

In a homogeneous atmosphere, the solutions are exponentials

$$\sim e^{i\vec{k}\cdot\vec{x}}\ e^{-i\omega t}$$

and sound waves have the following simple dispersion relationship

$$\omega^2 = k^2 c^2$$

Inertial Term

Pressure Gradient Force

ω Frequency
k Wavenumber
c Sound Speed

$$k^2 = \left|\vec{k}\right|^2$$

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Buoyancy Fluctuations and Internal Gravity Waves

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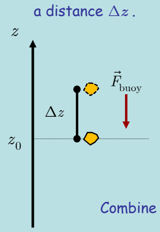
Buoyancy Fluctuations

Buoyancy is the combination of the gravitational force and the pressure support provided by the atmosphere. (Why does wood float?)

Let P and ρ be the pressure and density in the parcel and let P_0 and ρ_0 be those in the surrounding atmosphere. Lift a fluid parcel upwards by a distance Δz.



Adiabatic Motion $\quad \frac{DP}{Dt} = c^2 \frac{D\rho}{Dt} \quad \rightarrow \quad \delta P = c^2\, \delta\rho$ (Lagrangian Change)

Pressure Equilibration (Hydrostatic) $\quad \delta P = \Delta P_0 = -g\rho_0 \Delta z$

Combine to obtain

$$\delta\rho = -\frac{g\rho_0}{c^2}\Delta z \quad \text{Change in the parcel density}$$

$$\rho = \rho_0 + \delta\rho = \rho_0 - \frac{g\rho_0}{c^2}\Delta z \quad \text{Total density of the parcel}$$

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According to Archimedes' principle, the buoyancy force is equal to g times the density difference between the parcel and its new surroundings

$$F_{\rm buoy} = g\left\{\rho_0(z_0+\Delta z) - \left[\rho_0(z_0) + \delta\rho\right]\right\}$$

Background density at new position

Parcel density at new position

Expand the background density for small displacements

$$F_{\rm buoy} = g\left\{\left[\rho_0(z_0) + \frac{d\rho_0}{dz}\Delta z\right] - \left[\rho_0(z_0) + \delta\rho\right]\right\}$$

$$\delta\rho = -\frac{g\rho_0}{c^2}\Delta z$$

$$F_{\rm buoy} = g\left(\frac{d\rho_0}{dz} + \frac{g\rho_0}{c^2}\right)\Delta z$$

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Buoyancy Frequency

Express the force in terms of the acceleration

$$F_{\text{buoy}} = \rho_0 \frac{\partial^2 \Delta z}{\partial t^2} = g\left(\frac{d\rho_0}{dz} + \frac{g\rho_0}{c^2}\right)\Delta z$$

$$\frac{\partial^2 \Delta z}{\partial t^2} = -N^2 \Delta z$$

Buoyancy Frequency (or Brunt-Väisälä Frequency)

$$N^2 \equiv g\left(\frac{1}{H} - \frac{g}{c^2}\right)$$

Rate at which the atmospheric density changes ($\frac{1}{H}$)

Rate at which a fluid parcel's density changes ($\frac{g}{c^2}$)

Density Scale Height

$$H^{-1} \equiv -\frac{1}{\rho_0}\frac{d\rho_0}{dz}$$

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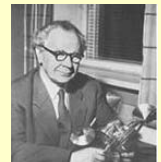
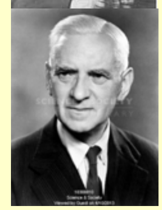
Brunt and Väisälä



Vilho Väisälä (1889–1969)

- Finnish mathematician, physicist and meterologist
- Inventor of meteorological instruments (balloon flights)



Sir David Brunt (1886–1965)

- Welsch meterologist
- Worked on dispersal of fog and other substances (including chemical warfare during the WWI)

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Internal Gravity Wave Dispersion

Stability Condition

$$\frac{\partial^2 \Delta z}{\partial t^2} = -N^2 \Delta z$$

$N^2 > 0$ Stable, oscillations

$N^2 < 0$ Unstable, convection

$$N^2 \equiv g\left(\frac{1}{H} - \frac{g}{c^2}\right)$$

Internal Gravity Wave Dispersion Relationship

$$\omega^2 = \frac{k_h^2}{k_z^2 + k_h^2} N^2$$

Inertial Term (ω^2)

Buoyancy Force

The wave frequency is always less than the buoyancy frequency.

Propagation requires horizontal variation.

Total Wavenumber

$$k^2 = k_z^2 + k_h^2$$

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Surface Gravity Waves

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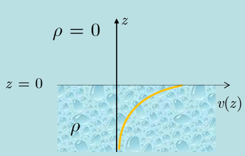
Surface Gravity Waves

In addition to internal gravity waves there is another family of gravity waves called surface gravity waves.

- Internal gravity waves - "reside" wherever $N^2 > 0$
- Surface gravity waves – "reside" on surfaces with rapid density change.



f mode (fundamental)

$$\sim e^{kz} e^{i(kx - \omega t)}$$

$$\omega^2 = gk$$

The *ocean-air interface* is one example and the *photosphere* is another.

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Part 2

Wave Equation for Acoustic-Gravity Waves

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Fluid Equations

We start from the fluid equations expressing the conservation of mass, momentum and energy. We ignore viscosity, thermal conduction, ionization, radiative transfer, and other nonadiabatic processes.

Momentum Equation
$$\rho \frac{D\vec{v}}{Dt} = -\vec{\nabla} P + \rho \vec{g}$$

Continuity Equation
$$\frac{D\rho}{Dt} = -\rho \vec{\nabla} \cdot \vec{v}$$

(Adiabatic) Energy Equation
$$\frac{DP}{Dt} = -\rho c^2 \vec{\nabla} \cdot \vec{v}$$

Both pressure and gravitation have been included as restoring forces

$\vec{g}$ = Gravitational Acceleration

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Linearization about a Steady Background Atmosphere

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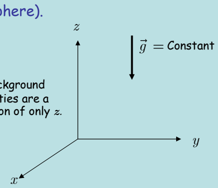
Linearize the Fluid Equations

Consider a vertically-stratified, compressible fluid in a uniform gravitational field (aka a plane-parallel atmosphere).

Linearize the variables about a steady state.

$$\rho(\vec{x},t) = \rho_0(z) + \rho_1(\vec{x},t)$$
$$P(\vec{x},t) = P_0(z) + P_1(\vec{x},t)$$
$$\vec{v}(\vec{x},t) = \qquad \vec{v}_1(\vec{x},t)$$

All background quantities are a function of only z.



Background Atmosphere

Hydrostatic Balance
$$\frac{dP_0}{dz} = -\rho_0 g$$

Note: The hydrostatic relation doesn't uniquely define the atmosphere. We also need the temperature profile $T = T(z)$ or, equivalently, $c = c(z)$.

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Fluctuation Equations

For convenience drop the "1" subscript from the velocity.

Continuity Equation
$$\frac{\partial \rho_1}{\partial t} + v_z \frac{d\rho_0}{dz} = -\rho_0 \vec{\nabla} \cdot \vec{v}$$

Horizontal Momentum Equation
$$\rho_0 \frac{\partial \vec{v}_h}{\partial t} = -\vec{\nabla}_h P_1$$

Vertical Momentum Equation
$$\rho_0 \frac{\partial v_z}{\partial t} = -\frac{\partial P_1}{\partial z} - g\rho_1$$

Energy Equation
$$\frac{\partial P_1}{\partial t} + v_z \frac{dP_0}{dz} = -\rho_0 c^2 \vec{\nabla} \cdot \vec{v}$$

where I have defined

$$\vec{v}_h \equiv v_x \hat{x} + v_y \hat{y}$$ Horizontal velocity

$$\vec{\nabla}_h \equiv \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y}$$ Horizontal gradient

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Invoke Hydrostatics

The pressure gradient appearing in the energy equation can be eliminated using the hydrostatic relation valid for the background atmosphere

Energy Equation
$$\frac{\partial P_1}{\partial t} + v_z \frac{dP_0}{dz} = -\rho_0 c^2 \vec{\nabla} \cdot \vec{v}$$

Hydrostatic Relation
$$\frac{dP_0}{dz} = -\rho_0 g$$

$$\frac{\partial P_1}{\partial t} = \rho_0 \left(g v_z - c^2 \vec{\nabla} \cdot \vec{v} \right)$$

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Eliminate the Fluctuating Density

$$\frac{1}{\rho_0} \frac{\partial}{\partial t} \left(\text{Horizontal Momentum} \right) \Rightarrow \frac{\partial^2 \vec{v}_h}{\partial t^2} = -\vec{\nabla}_h \frac{\partial}{\partial t} \left(\frac{P_1}{\rho_0} \right)$$

$$\frac{1}{\rho_0} \frac{\partial}{\partial t} \left(\text{Vertical Momentum} \right) \Rightarrow \frac{\partial^2 v_z}{\partial t^2} = -\frac{1}{\rho_0} \frac{\partial}{\partial z} \frac{\partial P_1}{\partial t} - \frac{g}{\rho_0} \frac{\partial \rho_1}{\partial t}$$

Insert the continuity equation to eliminate ρ_1.
$$\frac{\partial \rho_1}{\partial t} = -v_z \frac{d\rho_0}{dz} - \rho_0 \vec{\nabla} \cdot \vec{v}$$

$$\frac{\partial^2 v_z}{\partial t^2} = -\frac{1}{\rho_0} \frac{\partial}{\partial z} \frac{\partial P_1}{\partial t} + \frac{g}{\rho_0} \left(v_z \frac{\partial \rho_0}{\partial z} + \rho_0 \vec{\nabla} \cdot \vec{v} \right)$$

$$\frac{\partial^2 v_z}{\partial t^2} = -\frac{1}{\rho_0} \frac{\partial}{\partial z} \frac{\partial P_1}{\partial t} - \frac{g v_z}{H} + g \vec{\nabla} \cdot \vec{v}$$

Density Scale Height
$$H^{-1} \equiv -\frac{1}{\rho_0} \frac{d\rho_0}{dz}$$

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Derive a Buoyancy Force Equation

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Where's the Buoyancy?

We might expect that the buoyancy frequency should appear somewhere.

$$\frac{\partial^2 v_z}{\partial t^2} = -\frac{1}{\rho_0}\frac{\partial}{\partial z}\frac{\partial P_1}{\partial t} - \frac{g v_z}{H} + g\vec{\nabla}\cdot\vec{v}$$

Apply the product rule to "move" the density inside the vertical derivative

$$\frac{\partial^2 v_z}{\partial t^2} = -\left(\frac{\partial}{\partial z} - \frac{1}{H}\right)\frac{\partial}{\partial t}\left(\frac{P_1}{\rho_0}\right) - \frac{g v_z}{H} + g\vec{\nabla}\cdot\vec{v}$$

Insert the energy equation to eliminate P_1 in the term with $1/H$. $\frac{\partial P_1}{\partial t} = \rho_0\left(g v_z - c^2\vec{\nabla}\cdot\vec{v}\right)$

$$\frac{\partial^2 v_z}{\partial t^2} = -\frac{\partial}{\partial z}\frac{\partial}{\partial t}\left(\frac{P_1}{\rho_0}\right) + \frac{g v_z - c^2\vec{\nabla}\cdot\vec{v}}{H} - \frac{g v_z}{H} + g\vec{\nabla}\cdot\vec{v}$$

$$\frac{\partial^2 v_z}{\partial t^2} = -\frac{\partial}{\partial z}\frac{\partial}{\partial t}\left(\frac{P}{\rho}\right) - c^2\left(\frac{1}{H} - \frac{g}{c^2}\right)\vec{\nabla}\cdot\vec{v}$$

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There it is!

$$\frac{\partial^2 v_z}{\partial t^2} = -\frac{\partial}{\partial z}\frac{\partial}{\partial t}\left(\frac{P_1}{\rho_0}\right) - c^2\left(\frac{1}{H} - \frac{g}{c^2}\right)\vec{\nabla}\cdot\vec{v}$$

We can now "recognize" the term in blue as being proportional to the square of the buoyancy frequency.

$$\frac{\partial^2 v_z}{\partial t^2} = -\frac{\partial}{\partial z}\frac{\partial}{\partial t}\left(\frac{P_1}{\rho_0}\right) - \frac{N^2 c^2}{g}\vec{\nabla}\cdot\vec{v}$$

Buoyancy Frequency

$$N^2 \equiv g\left(\frac{1}{H} - \frac{g}{c^2}\right)$$

The buoyancy term is proportional to both

- the square of the buoyancy frequency (as expected)
- the dilation (or divergence of the velocity)

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Synopsis

We have reduced the original four equations to three. We have also written the vertical momentum equation such that gravitation and a portion of the pressure force have been combined into buoyancy.

Adiabaticity (1) $$\frac{\partial P_1}{\partial t} = \rho_0\left(g v_z - c^2\vec{\nabla}\cdot\vec{v}\right)$$

Horizontal Momentum $$\frac{\partial^2 \vec{v}_h}{\partial t^2} = -\vec{\nabla}_h\frac{\partial}{\partial t}\left(\frac{P_1}{\rho_0}\right)$$

Vertical Momentum $$\frac{\partial^2 v_z}{\partial t^2} = -\frac{\partial}{\partial z}\frac{\partial}{\partial t}\left(\frac{P_1}{\rho_0}\right) - \frac{N^2 c^2}{g}\vec{\nabla}\cdot\vec{v}$$

Produces Sound Waves

Produces Gravity Waves

Our goal at this point is to reduce these down to a single equation. Note, that the buoyancy force and the pressure force both depend on the dilation . . .

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Then, after a fair amount of work† . . .

†Derivation included at the end of this presentation

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Wave Equation

. . . one can derive a single ODE for the dilation

$$\chi = \vec{\nabla}\cdot\vec{v}$$

Since, the atmosphere is temporally steady and horizontally homogeneous (but vertically stratified), the solution can be written as a horizontal plane wave with the following form

$$\chi(\vec{x},t) = \chi(z)\; e^{i(\vec{k}_h\cdot\vec{x} - \omega t)}$$

Horizontal Wavenumber

$$\vec{k}_h \equiv k_x\hat{x} + k_y\hat{y}$$

2nd order ODE in the height

$$\left\{\frac{d^2}{dz^2} - \frac{1}{H}\frac{d}{dz} + \left[\frac{1}{H^2}\frac{dH}{dz} + \frac{\omega^2}{c^2} + k_h^2\left(\frac{N^2}{\omega^2} - 1\right)\right]\right\}\left(c^2\chi\right) = 0$$

$$N^2 \equiv g\left(\frac{1}{H} - \frac{g}{c^2}\right) \qquad H^{-1} \equiv -\frac{1}{\rho_0}\frac{d\rho_0}{dz}$$

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Other Wave Variables

All other wave variables can be derived from the dilation.

$$v_z = -\frac{\omega^2}{\omega^4 - g^2 k_h^2}\left[\frac{d}{dz} + \left(\frac{g k_h^2}{\omega^2} - \frac{1}{H}\right)\right]\left(c^2 \chi\right)$$

$$\vec{v}_h = \frac{i\vec{k}_h}{\omega^2}\left(g v_z - c^2 \chi\right)$$

$$P_1 = \frac{i\rho_0}{\omega}\left(g v_z - c^2 \chi\right)$$

$$\rho_1 = \frac{i\rho_0}{\omega}\left(\frac{v_z}{H} - \chi\right)$$

Note:
If the wave were to be incompressive, $\chi = 0$, there would only be a trivial solution unless

$$\omega^2 = g k_h$$

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The f mode

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Incompressive Waves?

If we are going to work in a higher order variable like the dilation, we had better see if there exists a non-trivial solution that has $\chi = 0$. Start with the three equations from our synopsis slide (#29).

$$\frac{\partial P_1}{\partial t} = \rho_0\left(g v_z - c^2 \vec{\nabla}\cdot\vec{v}\right) \Rightarrow \frac{\partial}{\partial t}\left(\frac{P_1}{\rho_0}\right) = g v_z$$

$$\frac{\partial^2 \vec{v}_h}{\partial t^2} = -\vec{\nabla}_h \frac{\partial}{\partial t}\left(\frac{P_1}{\rho_0}\right) \Rightarrow \frac{\partial^2 \vec{v}_h}{\partial t^2} = -g\vec{\nabla}_h v_z$$

$$\frac{\partial^2 v_z}{\partial t^2} = -\frac{\partial}{\partial z}\frac{\partial}{\partial t}\left(\frac{P_1}{\rho_0}\right) - \frac{N^2 c^2}{g}\vec{\nabla}\cdot\vec{v} \Rightarrow \frac{\partial^2 v_z}{\partial t^2} = -g\frac{\partial v_z}{\partial z}$$

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Horizontal Plane Waves

Since the atmosphere is temporally steady and horizontally homogeneous, we can assume horizontal plane waves

$$\vec{v}(\vec{x}, t) = \vec{v}(z)\; e^{i(\vec{k}_h\cdot\vec{x} - \omega t)}$$

$$\frac{\partial^2 \vec{v}_h}{\partial t^2} = -g\vec{\nabla}_h v_z \Rightarrow \vec{v}_h = i\frac{g\vec{k}_h}{\omega^2} v_z$$

$$\frac{\partial^2 v_z}{\partial t^2} = -g\frac{\partial v_z}{\partial z} \Rightarrow v_z = \frac{g}{\omega^2}\frac{\partial v_z}{\partial z}$$

90 degree phase difference

Elliptical motion?

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Dilation = Divergence

We have yet to use the definition of the dilation

$$\vec{\nabla}\cdot\vec{v} = i\vec{k}_h\cdot\vec{v}_h + \frac{\partial v_z}{\partial z} = 0$$

$$\left(-\frac{g k_h^2}{\omega^2} + \frac{\omega^2}{g}\right) v_z = 0 \Rightarrow \omega^4 - g^2 k_h^2 = 0$$

Use the dispersion relation

$$\omega^2 = g k_h$$

$$\vec{v}_h = i\frac{g\vec{k}_h}{\omega^2} v_z \rightarrow \vec{v}_h = i\hat{k}_h v_z$$

Circular Motion

$$v_z = \frac{g}{\omega^2}\frac{\partial v_z}{\partial z} \rightarrow v_z = e^{k_h z} e^{i(\vec{k}_h\cdot\vec{x} - \omega t)}$$

The solution we have been chasing down turns out to be a surface gravity wave or the f mode

$$\omega^2 = g k_h$$

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f mode (fundamental mode)

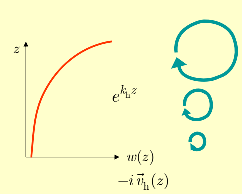
This wave (with zero dilation) **doesn't** depend on

- the sound speed $c(z)$ → Its **not** a sound wave
- or the buoyancy frequency $N(z)$ → Its **not** an internal gravity wave

The wave only depends on the gravitational acceleration and it lacks vertical nodes.

Therefore, this wave is

- Incompressive (zero dilation)
- A surface gravity wave (deep water wave)
- The *f* mode (**f**undamental)



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Part 3

Local Dispersion Relationship

2nd-Order ODE

Our ODE looks a bit daunting. But, remember it is just a 2nd order, albeit with non-constant coefficients.

$$\left\{\frac{d^2}{dz^2} - \frac{1}{H}\frac{d}{dz} + \left[\frac{1}{H^2}\frac{dH}{dz} + \frac{\omega^2}{c^2} + k_h^2\left(\frac{N^2}{\omega^2} - 1\right)\right]\right\}(c^2\chi) = 0$$

It looks less intimidating if we just realize that most of the complication is in the coefficient for the 0-order term.

$$\left\{\frac{d^2}{dz^2} - \frac{1}{H}\frac{d}{dz} + A(z)\right\}(c^2\chi) = 0$$

$$A(z) \equiv \frac{1}{H^2}\frac{dH}{dz} + \frac{\omega^2}{c^2} + k_h^2\left(\frac{N^2}{\omega^2} - 1\right)$$

Standard Form

We are used to seeing wave equations of the form

$$\frac{d^2\psi}{dz^2} + K^2\psi = 0 \qquad \text{Helmholtz Equation}$$

We can convert our ODE into this form by a change of variable. Let

$$c^2\chi = f(z)\psi(z)$$

Insert this expression into our ODE.

$$\frac{d^2\psi}{dz^2} + \left(\frac{2}{f}\frac{df}{dz} - \frac{1}{H}\right)\frac{d\psi}{dz} + \left[\frac{1}{f}\frac{d^2f}{dz^2} - \frac{1}{fH}\frac{df}{dz} + A(z)\right]\psi = 0$$

Choose f such that this term vanishes

Convert to a Helmholtz Equation

Choose the function f such that the coefficient in front of the first derivative is identically zero.

$$\frac{2}{f}\frac{df}{dz} - \frac{1}{H} = 0 \;\Rightarrow\; \frac{1}{f}\frac{df}{dz} = -\frac{1}{2}\left(\frac{1}{\rho_0}\frac{d\rho_0}{dz}\right) \;\Rightarrow\; \boxed{f = \rho_0^{-1/2}}$$

Then, our ODE takes on the desired form

$$\frac{d^2\psi}{dz^2} + k_z^2\psi = 0$$

with $$k_z^2(z) = \frac{1}{f}\frac{d^2f}{dz^2} - \frac{1}{fH}\frac{df}{dz} + A(z)$$

Wave Equation

Now insert the solution we found for f(z) to find

$$\chi = \left(c^2\rho_0^{1/2}\right)^{-1}\psi(z)\; e^{i\vec{k}_h\cdot\vec{x}}\; e^{-i\omega t}$$

$$\frac{d^2\psi}{dz^2} + k_z^2(z)\psi = 0 \qquad \text{Wave Equation (Helmholtz)}$$

$$k_z^2(z) = \frac{\omega^2 - \omega_c^2(z)}{c^2(z)} + k_h^2\left(\frac{N^2(z)}{\omega^2} - 1\right) \qquad \text{Local Vertical Wavenumber}$$

Buoyancy Frequency

$$N^2 = g\left(\frac{1}{H} - \frac{g}{c^2}\right)$$

Acoustic Cutoff Frequency

$$\omega_c^2 \equiv \frac{c^2}{4H^2}\left(1 - 2\frac{dH}{dz}\right)$$

Local Dispersion Relationship

$$k_z^2(z) = \frac{\omega^2 - \omega_c^2(z)}{c^2(z)} + k_h^2\left(\frac{N^2(z)}{\omega^2} - 1\right)$$

Pressure Fluctuations

Buoyancy Fluctuations

Density Stratification

Local Wavenumber
This is a dispersion relation that describes the *local* value of the vertical wavenumber. In general, the vertical wavenumber (or wavelength) will change with height in the atmosphere.

Acoustic-Gravity
Further, this relation describes both acoustic waves and internal gravity waves. In fact, the two are coupled into a single type of wave called an *acoustic-gravity wave*.

Two Restoring Forces
Acoustic-gravity waves have two restoring forces (pressure and buoyancy) that can work in concert or in opposition depending on the frequency and wavenumber.

Stars are Spheres!

If one is careful, the entire calculation can be performed in spherical coordinates. One finds that the same equation holds, with the radius replacing the height coordinate

$$\frac{d^2\psi}{dr^2} + k_r^2(r)\psi = 0$$

$$k_r^2(r) = \frac{\omega^2 - \omega_c^2(r)}{c^2(r)} + k_h^2(r)\left(\frac{N^2(r)}{\omega^2} - 1\right)$$

The only differences are that the horizontal wavenumber must take into account the spherical geometry

$$k_h^2(r) = \frac{l(l+1)}{r^2}$$

and . . .

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Sphericity Corrections

The characteristic frequencies have sphericity corrections

Buoyancy Frequency

$$N^2 = g\left(\frac{1}{\mathcal{H}} - \frac{g}{c^2} - \frac{2}{h}\right)$$

Acoustic Cutoff Frequency

$$\omega_c^2 = \frac{c^2}{4\mathcal{H}^2}\left(1 - 2\frac{d\mathcal{H}}{dr}\right) - \frac{g}{h}$$

A whole bunch of scale heights

$$\mathcal{H}^{-1} = H^{-1} + H_g^{-1} + H_f^{-1} + 3r^{-1}$$

$$h^{-1} = H_g^{-1} + 2r^{-1}$$

$$H_g^{-1} = -\frac{1}{g}\frac{dg}{dr}$$

$$H_f^{-1} = -\frac{d}{dr}\ln\left(2 + \frac{\omega^2 r}{g} + \frac{r}{H_g} - \frac{l(l+1)g}{\omega^2 r}\right)$$

Note:
Except for the horizontal wavenumber corrections, none of the corrections are important except near the very center of the star.

The corrections depend weakly on the frequency and on the harmonic degree l.

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Bonus Features

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Derivation of the Wave Equation

This additional material is for those of you who are mathematically minded and are unsatisfied when an instructor says, "It can be shown …"

I will provide a complete derivation for the wave equation for acoustic-gravity waves in a plane-parallel atmosphere with general stratification (a general temperature profile).

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Reminder

Fluctuation Equations

We have reduced the original four equations to three. We have also written the vertical momentum equation such that gravitation and a portion of the pressure force have been combined into buoyancy.

Adiabaticity (1) $$\frac{\partial P_1}{\partial t} = \rho_0\left(g v_z - c^2\vec{\nabla}\cdot\vec{v}\right)$$

Horizontal Momentum (2) $$\frac{\partial^2 \vec{v}_h}{\partial t^2} = -\vec{\nabla}_h\frac{\partial}{\partial t}\left(\frac{P_1}{\rho_0}\right)$$

Vertical Momentum (3) $$\frac{\partial^2 v_z}{\partial t^2} = -\frac{\partial}{\partial z}\frac{\partial}{\partial t}\left(\frac{P_1}{\rho_0}\right) - \frac{N^2c^2}{g}\vec{\nabla}\cdot\vec{v}$$

Our goal at this point is to reduce these down to a single equation. Note, that the buoyancy force and the pressure force both depend on the dilation . . .

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Eliminate the Pressure Fluctuation

Insert the energy equation (1) into the momentum equations (2) and (3) to eliminate the pressure fluctuation.

$$\frac{\partial^2 \vec{v}_\mathrm{h}}{\partial t^2} = -\vec{\nabla}_\mathrm{h} \frac{\partial}{\partial t}\left(\frac{P}{\bar{\rho}}\right)$$

$$\frac{\partial}{\partial t}\left(\frac{P}{\bar{\rho}}\right) = gw - c^2 \vec{\nabla}\cdot\vec{v}$$

$$\frac{\partial^2 \vec{v}_\mathrm{h}}{\partial t^2} = -\vec{\nabla}_\mathrm{h}\left(gw - c^2\vec{\nabla}\cdot\vec{v}\right) \quad (4)$$

$$\frac{\partial^2 w}{\partial t^2} = -\frac{\partial}{\partial z}\frac{\partial}{\partial t}\left(\frac{P}{\bar{\rho}}\right) - \frac{N^2 c^2}{g}\vec{\nabla}\cdot\vec{v}$$

$$\frac{\partial^2 w}{\partial t^2} = -g\frac{\partial w}{\partial z} + \left(\frac{\partial}{\partial z} - \frac{N^2}{g}\right)\left(c^2\vec{\nabla}\cdot\vec{v}\right) \quad (5)$$

Note:
Equations (4) and (5) form a set of coupled equations which are (in total) second order.

The dilation plays a major role in these equations (in both the pressure and buoyancy forces).

We will derive a single ODE in the dilation.

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Plane Wave Solution

Since the atmosphere is steady and horizontally invariant we can assume a plane wave solution

$$v_z(\vec{x},t) = v_z(z)\; e^{i\vec{k}_\mathrm{h}\cdot\vec{x}}\; e^{-i\omega t}$$

Horizontal Wavenumber

$$\vec{k}_\mathrm{h} = k_x\hat{x} + k_y\hat{y}$$

$$(4)\longrightarrow \quad \frac{\partial^2 \vec{v}_\mathrm{h}}{\partial t^2} = -\vec{\nabla}_\mathrm{h}\left(gv_z - c^2\vec{\nabla}\cdot\vec{v}\right)$$

$$-\omega^2\vec{v}_\mathrm{h} = -i\vec{k}_\mathrm{h}\left(gv_z - c^2\chi\right) \quad (6)$$

$$(5)\longrightarrow \quad \frac{\partial^2 v_z}{\partial t^2} = -g\frac{\partial v_z}{\partial z} + \left(\frac{\partial}{\partial z} - \frac{N^2}{g}\right)\left(c^2\vec{\nabla}\cdot\vec{v}\right)$$

$$-\omega^2 v_z = -g\frac{dv_z}{dz} + \left(\frac{d}{dz} - \frac{N^2}{g}\right)\left(c^2\chi\right) \quad (7)$$

Dilation

$$\chi \equiv \vec{\nabla}\cdot\vec{v}$$

$$\chi = i\vec{k}_\mathrm{h}\cdot\vec{v}_\mathrm{h} + \frac{dv_z}{dz}$$

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Find an equation for the dilation

Insert equations (6) and (7) into the definition for the divergence

$$(6)\longrightarrow \quad -\omega^2\vec{v}_\mathrm{h} = i\vec{k}_\mathrm{h}\left(c^2\chi - gv_z\right)$$

Horizontal Momentum Equation

$$(7)\longrightarrow \quad -\omega^2 v_z = -g\frac{dv_z}{dz} + \left(\frac{d}{dz} - \frac{N^2}{g}\right)\left(c^2\chi\right)$$

Vertical Momentum Equation

From (6) $\quad i\vec{k}_\mathrm{h}\cdot\vec{v}_\mathrm{h} = \frac{k_\mathrm{h}^2}{\omega^2}\left(c^2\chi - gv_z\right)$

$$\chi = i\vec{k}_\mathrm{h}\cdot\vec{v}_\mathrm{h} + \frac{dv_z}{dz} \quad \Rightarrow \quad \chi = \frac{k_\mathrm{h}^2}{\omega^2}\left(c^2\chi - gv_z\right) + \frac{dv_z}{dz} \quad (8)$$

Equations (7) and (8) are two independent, but coupled, equations in the variables v_z and χ. We can eliminate v_z by cross differentiation.

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Cross Differentiation

$$(7)\longrightarrow \quad -\omega^2 v_z = -g\frac{dv_z}{dz} + \left(\frac{d}{dz} - \frac{N^2}{g}\right)\left(c^2\chi\right)$$

$$\Rightarrow \left(g\frac{d}{dz} - \omega^2\right)v_z - \left(\frac{d}{dz} - \frac{N^2}{g}\right)\left(c^2\chi\right) = 0$$

$$\mathcal{L}_1 v_z - \mathcal{D}_1(c^2\chi) = 0$$

$$\mathcal{L}_1 \equiv \left(g\frac{d}{dz} - \omega^2\right) \qquad \mathcal{D}_1 \equiv \left(\frac{d}{dz} - \frac{N^2}{g}\right)$$

$$(8)\longrightarrow \quad \chi = \frac{k_\mathrm{h}^2}{\omega^2}\left(c^2\chi - gv_z\right) + \frac{dv_z}{dz}$$

$$\Rightarrow \left(\frac{d}{dz} - \frac{gk_\mathrm{h}^2}{\omega^2}\right)v_z - \left(\frac{\omega^2 - k_\mathrm{h}^2c^2}{\omega^2}\right)\chi = 0$$

$$\mathcal{L}_2 v_z - \mathcal{D}_2(c^2\chi) = 0$$

$$\mathcal{L}_2 \equiv \left(\frac{d}{dz} - \frac{gk_\mathrm{h}^2}{\omega^2}\right) \qquad \mathcal{D}_2 \equiv \left(\frac{\omega^2 - k_\mathrm{h}^2c^2}{\omega^2c^2}\right)$$

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Cross Differentiate and Subtract

$$\mathcal{L}_1\left[\mathcal{L}_2 v_z - \mathcal{D}_2(c^2\chi)\right] = 0$$

$$-\quad \mathcal{L}_2\left[\mathcal{L}_1 v_z - \mathcal{D}_1(c^2\chi)\right] = 0$$

$$\underbrace{\left[\mathcal{L}_1\mathcal{L}_2 - \mathcal{L}_2\mathcal{L}_1\right]}_{0} v_z - \left[\mathcal{L}_1\mathcal{D}_2 - \mathcal{L}_2\mathcal{D}_1\right]\left(c^2\chi\right) = 0$$

Note: Commutation!

$$\mathcal{L}_1\mathcal{L}_2 = \mathcal{L}_2\mathcal{L}_1$$

Simplify the terms to find a single equation for $c^2\chi$.

$$\left\{\frac{d^2}{dz^2} - \frac{1}{H}\frac{d}{dz} + \frac{1}{H^2}\frac{dH}{dz} + \frac{\omega^2}{c^2} + k_\mathrm{h}^2\left(\frac{N^2}{\omega^2} - 1\right)\right\}\left(c^2\chi\right) = 0 \quad (9)$$

Voila: We have a single ODE that describes the dilation.

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Synopsis

Therefore, we have a solution for the dilation that is a plane wave in the horizontal directions, and whose vertical behavior satisfies

$$\chi(\vec{x},t) = \chi(z)\; e^{i\vec{k}_\mathrm{h}\cdot\vec{x}}\; e^{-i\omega t}$$

$$\left\{\frac{d^2}{dz^2} - \frac{1}{H}\frac{d}{dz} + \frac{1}{H^2}\frac{dH}{dz} + \frac{\omega^2}{c^2} + k_\mathrm{h}^2\left(\frac{N^2}{\omega^2} - 1\right)\right\}\left(c^2\chi\right) = 0$$

All other properties can be derived from the dilation.

$$v_z = -\frac{\omega^2}{\omega^4 - g^2k_\mathrm{h}^2}\left[\frac{d}{dz} + \left(\frac{gk_\mathrm{h}^2}{\omega^2} - \frac{1}{H}\right)\right]\left(c^2\chi\right)$$

The vertical velocity can be obtained by eliminating dv_z/dz from equations (7) and (8).

$$\vec{v}_\mathrm{h} = \frac{i\vec{k}_\mathrm{h}}{\omega^2}\left(gv_z - c^2\chi\right) \qquad P_1 = \frac{i\rho_0}{\omega}\left(gv_z - c^2\chi\right) \qquad \rho_1 = \frac{i\rho_0}{\omega}\left(\frac{v_z}{H} - \chi\right)$$

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