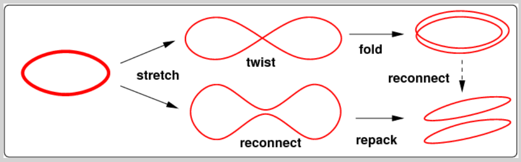
ASTR 7500: Solar & Stellar Magnetism

Hale CGEG Solar & Space Physics



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Lecture 10 Thurs 21 Feb 2013

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Dynamos: Motivation

- For $\mathbf{v} = 0$ magnetic field decays on timescale $\tau_d \sim L^2/\eta$
- Earth and other planets:
  - Evidence for magnetic field on earth for $3.5 \cdot 10^9$ years while $\tau_d \sim 10^4$ years
  - Permanent rock magnetism not possible since $T > T_{\rm Curie}$ and field highly variable $\longrightarrow$ field must be maintained by active process
- Sun and other stars:
  - Evidence for solar magnetic field for $\sim 300\,000$ years (^{10}Be)
  - Most solar-like stars show magnetic activity independent of age
  - Indirect evidence for stellar magnetic fields over life time of stars
  - But $\tau_d \sim 10^9$ years!
  - Primordial field could have survived in radiative interior of sun, but convection zone has much shorter diffusion time scale ~ 10 years (turbulent diffusivity)

Mathematical definition of dynamo

S bounded volume with the surface ∂S, $\mathbf{B}$ maintained by currents contained within S, $B \sim r^{-3}$ asymptotically,

$$\begin{aligned}
\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{v} \times \mathbf{B} - \eta \nabla \times \mathbf{B}) && \text{in } S \\
\nabla \times \mathbf{B} &= 0 && \text{outside } S \\
[\mathbf{B}] &= 0 && \text{across } \partial S \\
\nabla \cdot \mathbf{B} &= 0
\end{aligned}$$

$\mathbf{v} = 0$ outside S, $\mathbf{n} \cdot \mathbf{v} = 0$ on ∂S and

$$E_{\rm kin} = \int_S \frac{1}{2} \varrho \mathbf{v}^2 \, dV \le E_{\rm max} \quad \forall\, t$$

$\mathbf{v}$ is a dynamo if an initial condition $\mathbf{B} = \mathbf{B}_0$ exists so that

$$E_{\rm mag} = \int_{-\infty}^{\infty} \frac{1}{2\mu_0} \mathbf{B}^2 \, dV \ge E_{\rm min} \quad \forall\, t$$

Large scale/small scale dynamos

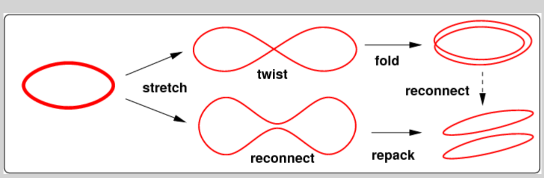
Decompose the magnetic field into large scale part and small scale part (energy carrying scale of turbulence) $\mathbf{B} = \overline{\mathbf{B}} + \mathbf{B}'$:

$$E_{\rm mag} = \int \frac{1}{2\mu_0} \overline{\mathbf{B}}^2 \, dV + \int \frac{1}{2\mu_0} \overline{\mathbf{B}'^2} \, dV \, .$$

- Small scale dynamo: $\overline{\mathbf{B}}^2 \ll \overline{\mathbf{B}'^2}$
- Large scale dynamo: $\overline{\mathbf{B}}^2 \ge \overline{\mathbf{B}'^2}$

Almost all turbulent (chaotic) velocity fields are small scale dynamos for sufficiently large R_m, large scale dynamos require additional large scale symmetries (see second half of this lecture)

Large scale/small scale dynamos



- Amplification through field line stretching
- Twist-fold required to repack field into original volume
- Twist-fold requires 3D - there are no dynamos is 2D!
- Magnetic diffusivity allows for change of topology

Slow/fast dynamos

Influence of magnetic diffusivity on growth rate

- Fast dynamo: growth rate independent of R_m (stretch-twist-fold mechanism)
- Slow dynamo: growth rate limited by resistivity (stretch-reconnect-repack)

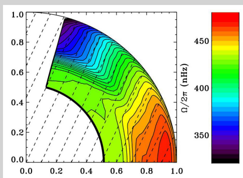
- Fast dynamos relevant for most astrophysical objects since $R_m \gg 1$
- Dynamos including (resistive) reconnection steps can be fast provided the reconnection is fast

Differential rotation and meridional flow

Induction effects of axisymmetric flows on axisymmetric field:

$$\begin{aligned}\mathbf{B} &= B\mathbf{e}_\Phi + \nabla\times(A\mathbf{e}_\Phi)\\ \mathbf{v} &= v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta + \Omega\, r\sin\theta\,\mathbf{e}_\Phi\end{aligned}$$

Differential rotation most dominant shear flow in stellar convection zones:



Meridional flow by-product of DR, observed as poleward surface flow in case of the sun

Differential rotation and meridional flow

Spherical geometry:

$$\frac{\partial B}{\partial t} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rv_rB)+\frac{\partial}{\partial\theta}(v_\theta B)\right) = r\sin\,\mathbf{B}_p\cdot\nabla\Omega + \eta\left(\Delta - \frac{1}{(r\sin\theta)^2}\right)B$$

$$\frac{\partial A}{\partial t} + \frac{1}{r\sin\theta}\mathbf{v}_p\cdot\nabla(r\sin\theta A) = \eta\left(\Delta-\frac{1}{(r\sin\theta)^2}\right)A$$

- Meridional flow: Independent advection of poloidal and toroidal field
- Differential rotation: Source for toroidal field (if poloidal field not zero)
- Diffusion: Sink for poloidal and toroidal field
- No term capable of maintaining poloidal field against Ohmic decay!

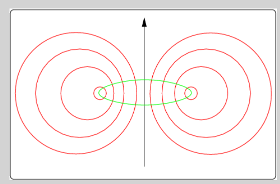
Differential rotation and meridional flow

- Weak poloidal seed field can lead to significant field amplification
- No source term for poloidal field
- Decay of poloidal field on resistive time scale
- Ultimate decay of toroidal field
- Not a dynamo!
- What is needed?

- Source for poloidal field

Cowling's anti-dynamo theorem

A stationary axisymmetric magnetic field with currents limited to a finite volume in space cannot be maintained by a velocity field with finite amplitude.



Ohm's law of the form $\mathbf{j}=\sigma\mathbf{E}$ only decaying solutions, focus here on $\mathbf{j}=\sigma(\mathbf{v}\times\mathbf{B})$.
On O-type neutral line $\mathbf{B}_p$ is zero, but $\mu_0\mathbf{j}_t = \nabla\times\mathbf{B}_p$ has finite value, but cannot be maintained by $(\mathbf{v}\times\mathbf{B})_t = (\mathbf{v}_p\times\mathbf{B}_p)$.

Large scale dynamo theory

Some history:

- 1919 Sir Joeseph Larmor: Solar magnetic field maintained by motions of conducting fluid?
- 1937 Cowling's anti-dynamo theorem and many others
- 1955 Parker: decomposition of field in axisymmetric and non-axisymmetric parts, average over induction effects of non-axisymmetric field
- 1964 Braginskii, Steenbeck, Krause: Mathematical frame work of mean field theory developed
- last 2 decades 3D dynamo simulations

Reynolds rules

We need to define an averaging procedure to define the mean and the fluctuating field.
For any function f and g decomposed as $f=\overline{f}+f'$ and $g=\overline{g}+g'$ we require that the Reynolds rules apply

$$\begin{aligned}\overline{\overline{f}} &= \overline{f} \longrightarrow \overline{f'}=0\\ \overline{f+g} &= \overline{f}+\overline{g}\\ \overline{\overline{f}g} &= \overline{f}\,\overline{g} \longrightarrow \overline{f'\overline{g}}=0\\ \overline{\partial f/\partial x_i} &= \partial\overline{f}/\partial x_i\\ \overline{\partial f/\partial t} &= \partial\overline{f}/\partial t\,.\end{aligned}$$

Examples:

- Longitudinal average (mean = axisymmetric component)
- Ensemble average (mean = average over several realizations of chaotic system)

Meanfield induction equation

Average of induction equation:

$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \nabla \times \left(\overline{\mathbf{v}' \times \mathbf{B}'} + \overline{\mathbf{v}} \times \overline{\mathbf{B}} - \eta \nabla \times \overline{\mathbf{B}} \right)$$

New term resulting from small scale effects:

$$\overline{\mathcal{E}} = \overline{\mathbf{v}' \times \mathbf{B}'}$$

Fluctuating part of induction equation:

$$\left(\frac{\partial}{\partial t} - \eta \Delta \right) \mathbf{B}' - \nabla \times (\overline{\mathbf{v}} \times \mathbf{B}') = \nabla \times \left(\mathbf{v}' \times \overline{\mathbf{B}} + \mathbf{v}' \times \mathbf{B}' - \overline{\mathbf{v}' \times \mathbf{B}'} \right)$$

Kinematic approach: $\mathbf{v}'$ assumed to be given

- Solve for $\mathbf{B}'$, compute $\overline{\mathbf{v}' \times \mathbf{B}'}$ and solve for $\overline{\mathbf{B}}$
- Term $\mathbf{v}' \times \mathbf{B}' - \overline{\mathbf{v}' \times \mathbf{B}'}$ leading to higher order correlations (closure problem)

Mean field expansion of turbulent induction effects

Exact expressions for $\overline{\mathcal{E}}$ exist only under strong simplifying assumptions (see homework assignment).

In general $\overline{\mathcal{E}}$ is a linear functional of $\overline{\mathbf{B}}$:

$$\overline{\mathcal{E}}_i(\mathbf{x}, t) = \int_{-\infty}^{\infty} d^3x' \int_{-\infty}^{t} dt' \, \mathcal{K}_{ij}(\mathbf{x}, t, \mathbf{x}', t') \, \overline{B}_j(\mathbf{x}', t') \, .$$

Can be simplified if a sufficient scale separation is present:

- $l_c \ll L$
- $\tau_c \ll \tau_L$

Leading terms of expansion:

$$\overline{\mathcal{E}}_i = a_{ij} \overline{B}_j + b_{ijk} \frac{\partial \overline{B}_j}{\partial x_k}$$

In stellar convection zones scale separation also only marginally justified (continuous turbulence spectrum)!

Symmetry constraints

Decomposing a_{ij} and $\partial \overline{B}_j / \partial x_k$ into symmetric and antisymmetric components:

$$a_{ij} = \underbrace{\frac{1}{2}(a_{ij} + a_{ji})}_{\alpha_{ij}} + \underbrace{\frac{1}{2}(a_{ij} - a_{ji})}_{-\varepsilon_{ijk}\gamma_k}$$

$$\frac{\partial \overline{B}_j}{\partial x_k} = \frac{1}{2}\left(\frac{\partial \overline{B}_j}{\partial x_k} + \frac{\partial \overline{B}_k}{\partial x_j} \right) + \underbrace{\frac{1}{2}\left(\frac{\partial \overline{B}_j}{\partial x_k} - \frac{\partial \overline{B}_k}{\partial x_j} \right)}_{-\frac{1}{2}\varepsilon_{jkl}(\nabla \times \overline{\mathbf{B}})_l}$$

Leads to:

$$\overline{\mathcal{E}}_i = \alpha_{ij} \overline{B}_j + \varepsilon_{ikj} \gamma_k \overline{B}_j - \underbrace{\frac{1}{2} b_{ijk} \varepsilon_{jkl}}_{\beta_{il} - \varepsilon_{ilm}\delta_m} (\nabla \times \overline{\mathbf{B}})_l + \ldots$$

Symmetry constraints

Overall result:

$$\overline{\mathcal{E}} = \boldsymbol{\alpha} \overline{\mathbf{B}} + \boldsymbol{\gamma} \times \overline{\mathbf{B}} - \boldsymbol{\beta} \, \nabla \times \overline{\mathbf{B}} - \boldsymbol{\delta} \times (\nabla \times \overline{\mathbf{B}}) + \ldots$$

With:

$$\alpha_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) \, , \qquad \gamma_i = -\frac{1}{2}\varepsilon_{ijk} a_{jk}$$

$$\beta_{ij} = \frac{1}{4}(\varepsilon_{ikl} b_{jkl} + \varepsilon_{jkl} b_{ikl}) \, , \qquad \delta_i = \frac{1}{4}(b_{jji} - b_{jij})$$

Mean field induction equation

Induction equation for $\overline{\mathbf{B}}$:

$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \nabla \times \left[\boldsymbol{\alpha} \overline{\mathbf{B}} + (\overline{\mathbf{v}} + \boldsymbol{\gamma}) \times \overline{\mathbf{B}} - (\eta + \boldsymbol{\beta}) \, \nabla \times \overline{\mathbf{B}} - \boldsymbol{\delta} \times (\nabla \times \overline{\mathbf{B}}) \right]$$

Interpretation on first sight:

- $\boldsymbol{\alpha}$: new effect
- $\boldsymbol{\gamma}$: acts like advection (turbulent advection effect)
- $\boldsymbol{\beta}$: acts like diffusion (turbulent diffusivity)
- $\boldsymbol{\delta}$: special anisotropy of diffusion tensor

Symmetry constraints

$\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ depend on large scale symmetries of the system defining the symmetry properties of the turbulence (e.g. rotation and stratification). Additional to that the expansion

$$\overline{\mathcal{E}} = \boldsymbol{\alpha} \overline{\mathbf{B}} + \boldsymbol{\gamma} \times \overline{\mathbf{B}} - \boldsymbol{\beta} \, \nabla \times \overline{\mathbf{B}} - \boldsymbol{\delta} \times \nabla \times \overline{\mathbf{B}} + \ldots$$

is a relation between polar and axial vectors:

- $\overline{\mathcal{E}}$: polar vector, independent from handedness of coordinate system
- $\overline{\mathbf{B}}$: axial vector, involves handedness of coordinate system in definition (curl operator, cross product)

Handedness of coordinate system pure convention (contains no physics), consistency requires:

- $\boldsymbol{\alpha}$, $\boldsymbol{\delta}$: pseudo tensor
- $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$: true tensors

Symmetry constraints

Turbulence with rotation and stratification

- true tensors: δ_{ij}, g_i, $g_i g_j$, $\Omega_i \Omega_j$, $\Omega_i \varepsilon_{ijk}$
- pseudo tensors: ε_{ijk}, Ω_i, $\Omega_i g_j$, $g_i \varepsilon_{ijk}$

Symmetry constraints allow only certain combinations:

$$\begin{aligned} \alpha_{ij} &= \alpha_0 (\mathbf{g}\cdot\boldsymbol{\Omega})\delta_{ij} + \alpha_1 (g_i\Omega_j + g_j\Omega_i) \ , \quad & \gamma_i &= \gamma_0 g_i + \gamma_1 \varepsilon_{ijk} g_j \Omega_k \\ \beta_{ij} &= \beta_0\,\delta_{ij} + \beta_1\, g_i g_j + \beta_2\,\Omega_i\Omega_j \ , & \delta_i &= \delta_0 \Omega_i \end{aligned}$$

The scalars $\alpha_0 \ldots \delta_0$ depend on quantities of the turbulence such as rms velocity and correlation times scale.

- isotropic turbulence: only $\boldsymbol{\beta}$
- + stratification: $\boldsymbol{\beta} + \boldsymbol{\gamma}$
- + rotation: $\boldsymbol{\beta} + \boldsymbol{\delta}$
- + stratification + rotation: $\boldsymbol{\alpha}$ can exist

Simplified expressions

Assuming $|\mathbf{B}'| \ll |\overline{\mathbf{B}}|$ in derivation + additional simplification for (quasi) isotropic, non-mirror symmetric, (weakly) inhomogeneous turbulence (see homework assignment):

$$\overline{v_i' v_j'} \sim \delta_{ij},\ \alpha_{ij} = \alpha\delta_{ij},\ \beta_{ij} = \eta_t \delta_{ij}$$

Leads to:

$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \boldsymbol{\nabla} \times \left[\alpha\overline{\mathbf{B}} + (\overline{\mathbf{v}} + \gamma) \times \overline{\mathbf{B}} - (\eta + \eta_t)\,\boldsymbol{\nabla} \times \overline{\mathbf{B}}\right]$$

with the scalar quantities

$$\alpha = -\frac{1}{3}\tau_c\,\overline{\mathbf{v}'\cdot(\boldsymbol{\nabla}\times\mathbf{v}')}, \qquad \eta_t = \frac{1}{3}\tau_c\,\overline{\mathbf{v}'^2}$$

and vector

$$\boldsymbol{\gamma} = -\frac{1}{6}\tau_c \boldsymbol{\nabla}\overline{\mathbf{v}'^2} = -\frac{1}{2}\boldsymbol{\nabla}\eta_t$$

Expressions are independent of η (in this approximation), indicating fast dynamo action!