Homework exercise: MHD dynamos

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1 Plasma Oscillations

The MHD approximation does not capture several high frequency plasma phenomena. The purpose of this excercise is to study plasma oscillations and the propagation of electromagnetic waves in a plasma. The generalized Ohm's law is given by:

$$\frac{\partial \boldsymbol{j}}{\partial t} + \frac{\boldsymbol{j}}{\tau_{ei}} = \frac{n_e q_e^2}{m_e}\boldsymbol{E} + \frac{q_e}{m_e}\boldsymbol{j} \times \boldsymbol{B} - \frac{q_e}{m_e}\boldsymbol{\nabla} p_e \tag{1}$$

For a non-magnetized ($\boldsymbol{B} = 0$), cool ($p_e = 0$) plasma and the high frequency limit ($\omega\tau_{ei} \gg 0$), Ohm's law takes the form:

$$\frac{\partial \boldsymbol{j}}{\partial t} = \frac{n_e q_e^2}{m_e}\boldsymbol{E} \tag{2}$$

Derive an equation describing plasma oscillation (electrons oscillate relative to ions in plasma) and determine their frequency (ω_p).

Derive an equation for the propagation of electromagnetic waves in a medium that is described by Eq. 2. Derive the dispersion relation as well as phase and group velocity. What happens for $\omega < \omega_p$? Estimate ω_p for the Earth's ionosphere. What are the consequences for radiowave propagation (think about communication, GPS).

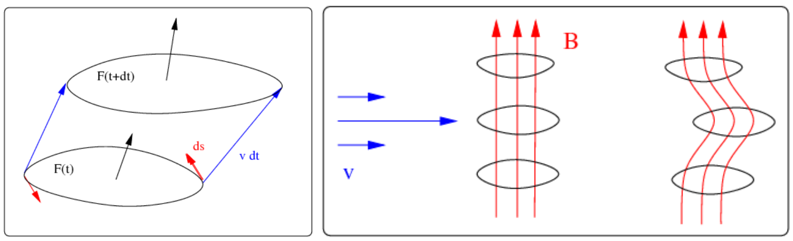
2 Non-relativistic Maxwell equations and field transformation

Start with the Maxwell equations and the transformation relation for charge density, current and electromagnetic field. Estimate the amplitude of terms in the Maxwell equations by approximating derivatives as $\nabla \sim l^{-1}$ and $\partial/\partial t \sim \tau^{-1}$. Assume $l \ll c\tau$ and neglect all terms of order $(l/c\tau)^2$. Assume high conductivity, i.e. the electric charge density of the plasma is purely due to dynamical effects. Derive the non-relativistic Galilei invariant Maxwell equations.

3 Alfvén's Theorem

Let Φ be the magnetic flux through a surface F with the property that its boundary ∂F is moving with the fluid:

$$\Phi = \int_F \boldsymbol{B} \cdot d\boldsymbol{f} \longrightarrow \frac{d\Phi}{dt} = 0 \tag{3}$$



Provide a proof of Alfvén's theorem. Tip: Use the ideal induction equation and the constraint $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$.

4 Turbulent induction effects

The purpose of this problem set is to derive expressions for the mean field electromotive force $\overline{\boldsymbol{\mathcal{E}}}$ under strongly simplifying assumptions. We start with a few problems in tensor algebra to recall a few mathematical skills useful for the following exercises.

4.1 Tensor algebra

For the following mathematical derivations it will be useful to use a component based notation for the manipulation of vector/tensor expressions. A term like $(\boldsymbol{A} \cdot \boldsymbol{\nabla})\boldsymbol{B}$ can be expressed as $A_k \partial B_i / \partial x_k$, where we use the "summation convention", which assumes that the duplication of the index "k" implies summation $k = 1, 2, 3$. Using the total antisymmetric Levi-Civita tensor ε_{ijk} we can express a cross product $\boldsymbol{A} \times \boldsymbol{B} = \boldsymbol{C}$ as $C_i = \varepsilon_{ijk} A_j B_k$. Note that ε_{ijk} is +1 for even perturbations of (1,2,3), -1 for odd perturbations of (1,2,3) and 0 otherwise. A useful relation for expressions including products of the Levi-Civita tensor is the identity ("contracted epsilon identity")

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \tag{4}$$

with the Kronecker symbol δ_{ik}, which is 1 for $i = j$ and 0 otherwise.

Problems:

a) Compute the double contraction $\varepsilon_{ijk}\varepsilon_{ijl}$.

b) Proof the vector identity

$$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = -(\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + \boldsymbol{A}\nabla \cdot \boldsymbol{B} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - \boldsymbol{B}\nabla \cdot \boldsymbol{A} \,. \quad (5)$$

c) Any anti-symmetric tensor, $a_{ij} = -a_{ji}$, has three independent components (i.e. the elements above the diagonal). It can therefore be expressed in terms of a 3-component vector using the Levi-Civita symbol, $a_{ij} = -\epsilon_{ijk}\gamma_k$. Derive an inverse expression given the vector γ_k explicitly in terms of a_{ij}.

4.2 Second order correlation approximation

Problems:

a) Start from the induction equation for $\boldsymbol{B}'$ (Volume I, Eq. 3.44):

$$\frac{\partial \boldsymbol{B}'}{\partial t} = \nabla \times \left(\boldsymbol{v}' \times \overline{\boldsymbol{B}} + \overline{\boldsymbol{v}} \times \boldsymbol{B}' - \eta\nabla \times \boldsymbol{B}' + \boldsymbol{v}' \times \boldsymbol{B}' - \overline{\boldsymbol{v}' \times \boldsymbol{B}'}\right) \,, \quad (6)$$

and assume $\overline{\boldsymbol{v}} = 0$, $|\boldsymbol{B}'| \ll |\overline{\boldsymbol{B}}|$ and neglect the contribution from magnetic resistivity. Formally integrate the equation to obtain a solution for $\boldsymbol{B}'$ and derive an expression for $\overline{\boldsymbol{\mathcal{E}}} = \overline{\boldsymbol{v}' \times \boldsymbol{B}'}$. Assume that $\boldsymbol{v}'$ has a finite correlation time, τ_c, and simplify expressions by approximating time integrals with $\int_{-\infty}^{t} \overline{v_i'(t)v_k'(s)}\mathrm{d}s = \tau_c\overline{v_i'(t)v_k'(t)}$.

b) Express now all terms using the component notation summarized in Sect. 4.1 and show that the tensors a_{ij} and b_{ijk} in the expansion $\overline{\mathcal{E}}_i = a_{ij}\overline{B}_j + b_{ijk}\partial\overline{B}_j/\partial x_k$ are given by:

$$a_{ij} = \tau_c\overline{\left(\varepsilon_{ikl}v_k'\frac{\partial v_l'}{\partial x_j} - \varepsilon_{ikj}v_k'\frac{\partial v_m'}{\partial x_m}\right)} \quad (7)$$

$$b_{ijk} = \tau_c\varepsilon_{ijm}\overline{v_m'v_k'} \,. \quad (8)$$

c) Decompose these tensors into the terms $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ defined through:

$$\alpha_{ij} = \frac{1}{2}\left(a_{ij} + a_{ji}\right)$$

$$\gamma_i = -\frac{1}{2}\varepsilon_{ijk}a_{jk}$$

$$\beta_{ij} = \frac{1}{4}\left(\varepsilon_{ikl}b_{jkl} + \varepsilon_{jkl}b_{ikl}\right) \,.$$

Compute the trace α_{ii} and β_{ii}. To which physical quantities are they related?

d) Make now the additional assumption of isotropy, which implies that α_{ij}, β_{ij}, as well as the correlation tensor $\overline{v_i'v_j'}$ are diagonal, i.e. $\alpha_{ij} = \alpha\delta_{ij}$. Compute the scalar α-effect and the turbulent diffusivity η_t. How is $\boldsymbol{\gamma}$ related to η_t? Discuss under which conditions these effects exist.

5 Magnetic helicity and dynamos

Problems:

a) Start with the non-ideal MHD induction equation and derive an equation for the time evolution of the magnetic helicity $\boldsymbol{A} \cdot \boldsymbol{B}$, where $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$.

b) Repeat a) starting from the meanfield induction equation and derive an equation for the evolution of the meanfield helicity $\overline{\boldsymbol{A}} \cdot \overline{\boldsymbol{B}}$.

c) Derive the equation for the evolution of the small scale helicity $\boldsymbol{A}' \cdot \boldsymbol{B}'$ by subtracting the former two equations.

6 "Biermann battery"

The MHD induction equation is linear in $\boldsymbol{B}$, which implies that a dynamo cannot produce magnetic field if the initial condition was $\boldsymbol{B} = 0$. Start from the more general form of Ohm's law and keep the electron pressure term. Re-derive the induction equation and discuss under which conditions the additional term can act as an inhomogeneous source term independent of $\boldsymbol{B}$. Discuss the similarity with the vorticity equation in hydrodynamics. Describe situations in which this term could produce magnetic seed fields.