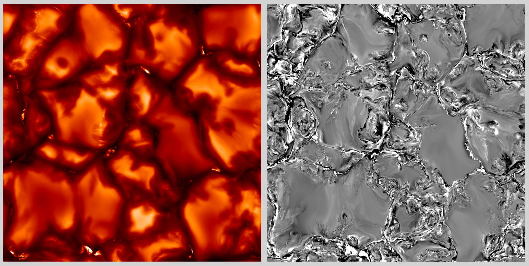
PHYS 7810 Special Topics in Physics : Physics of the Solar Atmosphere



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Spring 2019

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Scope of this lecture

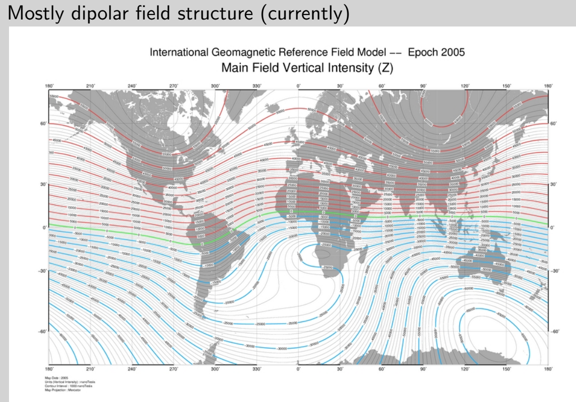
- Processes of magnetic field generation and destruction in turbulent plasma flows
- Introduction to general concepts of dynamo theory (this is not a lecture about the solar dynamo!)
- Outline
  - Intro: Magnetic fields in the Universe
  - MHD, induction equation
  - Some general remarks and definitions regarding dynamos
  - Small scale dynamos
  - Large scale dynamos (mean field theory)
    - Kinematic theory
    - Characterization of possible dynamos
    - Non-kinematic effects
  - Concluding remarks

Magnetic fields in the Universe

- Earth
  - Field strength ~ 0.5G
  - Magnetic field present for $\sim 3.5 \cdot 10^9$ years, much longer than Ohmic decay time ($\sim 10^4$ years)
  - Strong variability on shorter time scales (10^3 years)
- Mercury, Ganymede, (Io), Jupiter, Saturn, Uranus, Neptune have large scale fields
- Sun
  - Magnetic fields from smallest observable scales to size of sun
  - 22 year cycle of large scale field
  - Ohmic decay time $\sim 10^9$ years (in absence of turbulence)
- Other stars
  - Stars with outer convection zone: similar to sun
  - Stars with outer radiation zone: primordial fields, field generation in convective core
- Galaxies
  - Field strength $\sim \mu$G
  - Field structure coupled to observed matter distribution

Geomagnetism

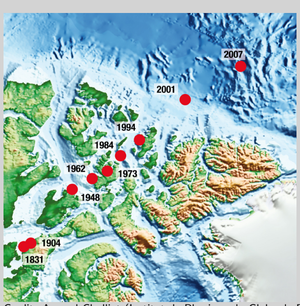
Mostly dipolar field structure (currently)



Credit: NOAA NGDC

Geomagnetism

Short-term variation on scales of hundreds of years

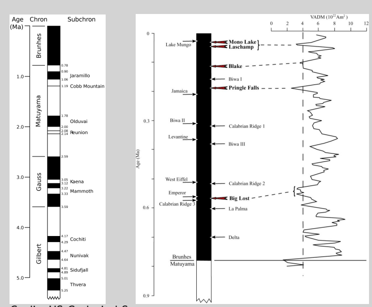


Credit: Arnaud Chulliat (Institut de Physique du Globe de Paris)

- Independent movement of the poles
- South and North pole are in general not opposite to each other (higher multipoles)
- Movements up to 40 km/year ($\sim$ 1 mm/sec)

Geomagnetism

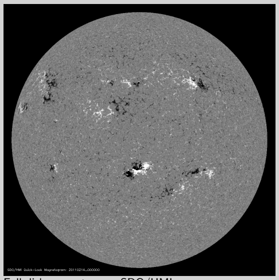
Long-term variation on scales of thousands to millions of years (deduced from volcanic rocks and sediments)



Credit: US Geological Survey

- Mostly random changes of polarity
- A given polarity for $\sim 100,000$ years
- Fast switches ~ 1000 years
- Strong variation of dipole moment and failed reversals

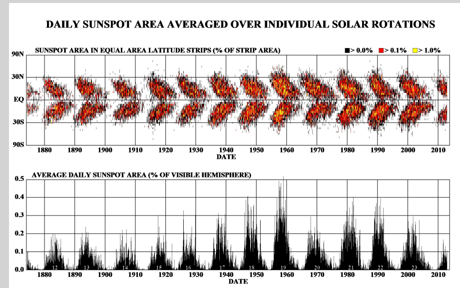
Solar Magnetism



Full disk magnetogram SDO/HMI

- Up to 4kG (sunspot umbra) field in solar photosphere
- Structured over the full range of observable scales from 100 km to size of Sun
- Large scale field shows symmetries with respect to equator

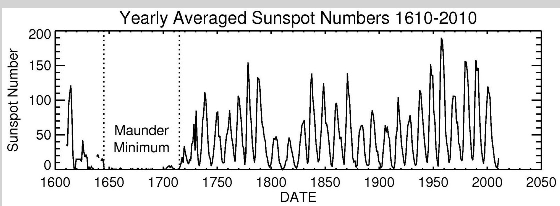
Solar Magnetism



Movie

- Large scale field exhibits ~ 22 year magnetic cycle
- 11 year cycle present in large scale flow variations (meridional flow and differential rotation)

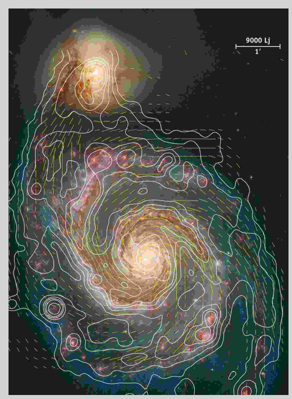
Solar Magnetism



Credit: NASA

- Cycle interrupted by grand minima with duration of up to 100 years
- Similar overall activity has been present for past $\sim 100,000$ years (tree ring and ice core records of cosmogenic isotopes: C-14 and Be-10).

Galactic magnetism



M51, Credit: MPI for Radioastronomy, Germany

- Magnetic field derived from polarization of radio emission
- μG field strength
- Magnetic field follows spiral structure to some extent
- Optically thin dynamo - Dynamo region can be observed!

Magnetic fields in the Universe

- Objects from size of a planet to galaxy clusters have large scale ($\sim$ size of object) magnetic fields
- Physical properties of object differ substantially
  - 1,000 km to 100,000 LJ
  - liquid iron to partially ionized plasma
  - spherical to disk-shaped
  - varying influence of rotation (but all of them are rotating)
  - $R_m \sim 10^3 \ldots 10^{18}$
  - ....
- Is there a common origin of magnetic field in these objects?
- Can we understand this on basis of MHD?

MHD equations

The basic framework for understanding the dynamics of a magnetized fluid are the MHD equations. In their most simple form they are applicable under the following conditions:

- Validity of continuum approximation (enough particles to define averages)
- Strong collisional coupling: validity of single fluid approximations, isotropic (scalar) gas pressure
- Non-relativistic motions, low frequencies, high electrical conductivity

They combine a fluid description in terms of the Navier-Stokes equations with the non-relativistic Maxwell equations as well as Ohm's Law.

Convective derivative

Continuum approximation means that density (ϱ), velocity ($\boldsymbol{v}$) and internal energy (e) can be written a functions of space ($\boldsymbol{x}$) and time (t). Quantities can vary in time (along a fluid trajectory) by either having in-stationary fields (time variation at a fixed location) or by having transport of a quantity in an inhomogeneous field (variation in space), i.e.

$$\begin{aligned}\frac{d\varrho}{dt} &= \lim_{\Delta t \to 0} \frac{\varrho(\boldsymbol{x}+\boldsymbol{v}\Delta t, t+\Delta t) - \varrho(\boldsymbol{x}, t)}{\Delta t} \\ &= \frac{\partial \varrho}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \varrho\end{aligned}$$

Equation of continuity

Mass density changes along fluid trajectory if flow is compressible ($\nabla \cdot \boldsymbol{v} \neq 0$):

$$\frac{d\varrho}{dt} = \frac{\partial \varrho}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \varrho = -\varrho \boldsymbol{\nabla} \cdot \boldsymbol{v}$$

Equivalent conservative formulation:

$$\frac{\partial \varrho}{\partial t} + \boldsymbol{\nabla} \cdot (\varrho \boldsymbol{v}) = 0$$

Change of mass in a fixed control volume Ω is given by mass flux across boundary $\partial\Omega$:

$$\frac{\partial m}{\partial t} = \int_\Omega \frac{\partial \varrho}{\partial t} dV = -\int_\Omega \boldsymbol{\nabla} \cdot (\varrho \boldsymbol{v}) dV = -\int_{\partial\Omega} (\varrho \boldsymbol{v}) \boldsymbol{n} dA$$

Momentum equation

Starting with Newton's law $m\, d\boldsymbol{v}/dt = \boldsymbol{F}$ we get per volume:

$$\varrho \frac{d\boldsymbol{v}}{dt} = \varrho \frac{\partial \boldsymbol{v}}{\partial t} + \varrho(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = \boldsymbol{f} = \underbrace{-\boldsymbol{\nabla} p + \varrho \boldsymbol{g}}_{\text{Pressure/Buoyancy force}} + \underbrace{\boldsymbol{\nabla} \cdot \bar{\tau}}_{\text{Viscous force}}$$

Using the equation of continuity:

$$\varrho \frac{\partial \boldsymbol{v}}{\partial t} + \varrho(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = \frac{\partial(\varrho \boldsymbol{v})}{\partial t} + \boldsymbol{v} \boldsymbol{\nabla} \cdot (\varrho \boldsymbol{v}) + (\varrho \boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = \frac{\partial(\varrho \boldsymbol{v})}{\partial t} + \boldsymbol{\nabla} \cdot (\varrho \boldsymbol{v}\boldsymbol{v})$$

We can derive the "conservative" form of the momentum equation:

$$\frac{\partial(\varrho \boldsymbol{v})}{\partial t} + \boldsymbol{\nabla} \cdot [\varrho \boldsymbol{v}\boldsymbol{v} + p\bar{\boldsymbol{I}} - \bar{\tau}] = \varrho \boldsymbol{g}$$

where $\boldsymbol{v}\boldsymbol{v} = v_i v_k$ denotes the dyadic product.

Kinetic energy equation

With the identity $(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = \boldsymbol{\nabla} \frac{v^2}{2} - \boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{v})$ follows:

$$\varrho \frac{\partial}{\partial t} \frac{v^2}{2} + \varrho \boldsymbol{v} \cdot \boldsymbol{\nabla} \frac{v^2}{2} = -\boldsymbol{v} \cdot \boldsymbol{\nabla} p + \varrho \boldsymbol{v} \cdot \boldsymbol{g} + \boldsymbol{v} \boldsymbol{\nabla} \cdot \bar{\tau}$$

Using again the equation of continuity can derive a "conservative" form:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\varrho v^2\right) + \boldsymbol{\nabla} \cdot \left[\boldsymbol{v}\left(\frac{1}{2}\varrho v^2 + p\right) - \boldsymbol{v}\bar{\tau}\right] = p\boldsymbol{\nabla} \cdot \boldsymbol{v} + \varrho \boldsymbol{v} \cdot \boldsymbol{g} - \underbrace{\bar{\tau} \cdot \boldsymbol{\nabla} \boldsymbol{v}}_{Q_\nu}$$

Here $Q_\nu = \bar{\tau} \cdot \boldsymbol{\nabla} \boldsymbol{v} = \tau_{ik} \partial_k v_i$ is the formal expression for the viscous heating.

Internal energy and total energy equation

First law of thermodynamics: $dE + pdV = dW$. With $e = E/m$ and $Q = V^{-1}\, dW/dt$ we can write:

$$\varrho \frac{de}{dt} - \frac{p}{\varrho} \frac{d\varrho}{dt} = Q$$

With the equation of continuity and $Q = Q_\nu + \boldsymbol{\nabla} \cdot (\kappa \boldsymbol{\nabla} T)$ (κ heat conductivity) follows:

$$\frac{\partial}{\partial t}(\varrho e) + \boldsymbol{\nabla} \cdot (\boldsymbol{v} \varrho e - \kappa \boldsymbol{\nabla} T) = -p\boldsymbol{\nabla} \cdot \boldsymbol{v} + Q_\nu$$

Combination with the kinetic energy equation yields:

$$\frac{\partial}{\partial t}\left(\varrho e + \frac{1}{2}\varrho v^2\right) + \boldsymbol{\nabla} \cdot \left[\boldsymbol{v}\left(\varrho e + \frac{1}{2}\varrho v^2 + p\right) - \boldsymbol{v}\bar{\tau} - \kappa \boldsymbol{\nabla} T\right] = \varrho \boldsymbol{v} \cdot \boldsymbol{g}$$

Navier-Stokes Equations

Primitive formulation:

$$\begin{aligned}\frac{\partial \varrho}{\partial t} &= -\boldsymbol{\nabla} \cdot (\varrho \boldsymbol{v}) \\ \varrho \frac{\partial \boldsymbol{v}}{\partial t} &= -\varrho(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} - \boldsymbol{\nabla} p + \varrho \boldsymbol{g} + \boldsymbol{\nabla} \cdot \bar{\tau} \\ \varrho \frac{\partial e}{\partial t} &= -\varrho(\boldsymbol{v} \cdot \boldsymbol{\nabla})e - p\boldsymbol{\nabla} \cdot \boldsymbol{v} + \boldsymbol{\nabla} \cdot (\kappa \boldsymbol{\nabla} T) + Q_\nu\end{aligned}$$

Conservative formulation ($E_{HD} = \varrho e + \frac{1}{2}\varrho v^2$):

$$\begin{aligned}\frac{\partial \varrho}{\partial t} + \boldsymbol{\nabla} \cdot (\varrho \boldsymbol{v}) &= 0 \\ \frac{\partial(\varrho \boldsymbol{v})}{\partial t} + \boldsymbol{\nabla} \cdot [\varrho \boldsymbol{v}\boldsymbol{v} + p\bar{\boldsymbol{I}} - \bar{\tau}] &= \varrho \boldsymbol{g} \\ \frac{\partial}{\partial t} E_{HD} + \boldsymbol{\nabla} \cdot [\boldsymbol{v}(E_{HD} + p) - \boldsymbol{v}\bar{\tau} - \kappa \boldsymbol{\nabla} T] &= \varrho \boldsymbol{v} \cdot \boldsymbol{g}\end{aligned}$$

Navier-Stokes Equations

Viscous stress tensor $\bar{\bar{\tau}}$

$$\Lambda_{ik} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)$$
$$\tau_{ik} = 2\varrho\nu\left(\Lambda_{ik} - \frac{1}{3}\delta_{ik}\nabla\cdot\boldsymbol{v}\right)$$
$$Q_\nu = \tau_{ik}\Lambda_{ik}\ ,$$

Equation of state (ideal gas)

$$p = (\gamma - 1)\varrho\, e\ .$$

ν and κ: viscosity and thermal conductivity

Maxwell Equations (relativistic)

Maxwell Equations

$$\nabla\cdot\boldsymbol{E} = \frac{\lambda}{\varepsilon_0} \qquad \nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}$$
$$\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{j} + \frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t} \qquad \nabla\cdot\boldsymbol{B} = 0$$

Transformation of the Lorentz vectors $(ct, \boldsymbol{x})$ and $(c\lambda, \boldsymbol{j})$ $(\gamma = \sqrt{1 - v^2/c^2}^{-1})$:

$$t' = \gamma\left(t - \frac{\boldsymbol{v}\cdot\boldsymbol{x}}{c^2}\right) \qquad \boldsymbol{x}' = \gamma\,(\boldsymbol{x} - \boldsymbol{v}\,t)$$
$$\lambda' = \gamma\left(\lambda - \frac{\boldsymbol{v}\cdot\boldsymbol{j}}{c^2}\right) \qquad \boldsymbol{j}' = \gamma\,(\boldsymbol{j} - \boldsymbol{v}\,\lambda)$$

Transformation of the electro-magnetic field:

$$\boldsymbol{E}'_\parallel = \boldsymbol{E}_\parallel \qquad \boldsymbol{B}'_\parallel = \boldsymbol{B}_\parallel$$
$$\boldsymbol{E}'_\perp = \gamma\,(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})_\perp \qquad \boldsymbol{B}'_\perp = \gamma\left(\boldsymbol{B} + \frac{1}{c^2}\boldsymbol{v}\times\boldsymbol{E}\right)_\perp$$

Maxwell Equations (non-relativistic)

Assumptions:

- non-relativistic motions and slow evolutions
- high conductivity

$\longrightarrow E \ll cB, v\lambda \ll j, \partial E/\partial t \ll c^2\,\mu_0 j$

$$\nabla\cdot\boldsymbol{E} = \frac{\lambda}{\varepsilon_0} \qquad \nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}$$
$$\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{j} \qquad \nabla\cdot\boldsymbol{B} = 0$$

Simplified transformations (Galilei-transformation):

$$t' = t \qquad \boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{v}\,t$$
$$\lambda' = \lambda - \frac{\boldsymbol{v}\cdot\boldsymbol{j}}{c^2} \qquad \boldsymbol{j}' = \boldsymbol{j}$$
$$\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B} \qquad \boldsymbol{B}' = \boldsymbol{B}$$

Ohm's law

Equation of motion for drift velocity $\boldsymbol{v}_d$ of electrons

$$n_e m_e\left(\frac{\partial v_d}{\partial t} + \frac{v_d}{\tau_{ei}}\right) = n_e q_e(\boldsymbol{E} + \boldsymbol{v}_d\times\boldsymbol{B}) - \nabla p_e$$

τ_{ei}: collision time between electrons and ions
n_e: electron density
q_e: electron charge
m_e: electron mass
p_e: electron pressure
With the electric current: $\boldsymbol{j} = n_e\,q_e\,\boldsymbol{v}_d$ this gives the generalized Ohm's law:

$$\frac{\partial\boldsymbol{j}}{\partial t} + \frac{\boldsymbol{j}}{\tau_{ei}} = \frac{n_e q_e^2}{m_e}\boldsymbol{E} + \frac{q_e}{m_e}\boldsymbol{j}\times\boldsymbol{B} - \frac{q_e}{m_e}\nabla p_e$$

Simplifications:

- $\tau_{ei}\,\omega_L \ll 1$, $\omega_L = eB/m_e$: Larmor frequency
- neglect ∇p_e
- low frequencies (no plasma oscillations)

Ohm's law

Simplified Ohm's law

$$\boldsymbol{j} = \sigma\boldsymbol{E}$$

with the plasma conductivity

$$\sigma = \frac{\tau_{ei}\,n_e q_e^2}{m_e}$$

The Ohm's law we derived so far is only valid in the co-moving frame of the plasma. Under the assumption of non-relativistic motions this transforms in the laboratory frame to

$$\boldsymbol{j} = \sigma\,(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})$$

Induction equation

Using Ampere's law $\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{j}$ yields for the electric field in the laboratory frame

$$\boldsymbol{E} = -\boldsymbol{v}\times\boldsymbol{B} + \frac{1}{\mu_0\sigma}\nabla\times\boldsymbol{B}$$

leading to the induction equation

$$\frac{\partial\boldsymbol{B}}{\partial t} = -\nabla\times\boldsymbol{E} = \nabla\times(\boldsymbol{v}\times\boldsymbol{B} - \eta\,\nabla\times\boldsymbol{B})$$

with the magnetic diffusivity

$$\eta = \frac{1}{\mu_0\sigma}\ .$$

MHD equations

The full set of MHD equations combines the induction equation with the Navier-Stokes equations including the Lorentz-force:

$$
\begin{aligned}
\frac{\partial \varrho}{\partial t} &= -\nabla \cdot (\varrho \boldsymbol{v}) \\
\varrho \frac{\partial \boldsymbol{v}}{\partial t} &= -\varrho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nabla p + \varrho \boldsymbol{g} + \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \nabla \cdot \bar{\boldsymbol{\tau}} \\
\varrho \frac{\partial e}{\partial t} &= -\varrho(\boldsymbol{v}\cdot\nabla)e - p\nabla\cdot\boldsymbol{v} + \nabla\cdot(\kappa\nabla T) + Q_\nu + Q_\eta \\
\frac{\partial \boldsymbol{B}}{\partial t} &= \nabla \times (\boldsymbol{v} \times \boldsymbol{B} - \eta \nabla \times \boldsymbol{B})
\end{aligned}
$$

Assumptions:

- Validity of continuum approximation (enough particles to define averages)
- Non-relativistic motions, low frequencies
- Strong collisional coupling: validity of single fluid approximations, isotropic (scalar) gas pressure

MHD equations

Viscous stress tensor τ

$$
\begin{aligned}
\Lambda_{ik} &= \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) \\
\tau_{ik} &= 2\varrho\nu\left(\Lambda_{ik} - \frac{1}{3}\delta_{ik}\nabla\cdot\boldsymbol{v}\right) \\
Q_\nu &= \tau_{ik}\Lambda_{ik} \,,
\end{aligned}
$$

Ohmic dissipation Q_η

$$Q_\eta = \frac{\eta}{\mu_0}(\nabla \times \boldsymbol{B})^2 \,.$$

Equation of state

$$p = (\gamma - 1)\varrho\, e \,.$$

ν, η and κ: viscosity, magnetic diffusivity and thermal conductivity
μ_0 denotes the permeability of vacuum