Kinematic approach

- Solving the 3D MHD equations is not always feasible
- Semi-analytical approach preferred for understanding fundamental properties of dynamos
- Evaluate turbulent induction effects based on induction equation for a given velocity field
  - Velocity field assumed to be given as 'background' turbulence, Lorentz-force feedback neglected (sufficiently weak magnetic field)
  - What correlations of a turbulent velocity field are required for dynamo (large scale) action?
  - Theory of onset of dynamo action, but not for non-linear saturation
- More detailed discussion of induction equation

Advection, diffusion, magnetic Reynolds number

L: typical length scale U: typical velocity scale L/U: time unit

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times \left(\boldsymbol{v} \times \boldsymbol{B} - \frac{1}{R_m} \boldsymbol{\nabla} \times \boldsymbol{B} \right)$$

with the magnetic Reynolds number

$$R_m = \frac{U\,L}{\eta} \,.$$

Advection, diffusion, magnetic Reynolds number

$R_m \ll 1$: diffusion dominated regime

$$\frac{\partial \boldsymbol{B}}{\partial t} = \eta \Delta \boldsymbol{B} \,.$$

Only decaying solutions with decay (diffusion) time scale

$$\tau_{\rm d} \sim \frac{L^2}{\eta}$$

Object	$\eta[\mathrm{m}^2/\mathrm{s}]$	$L[\mathrm{m}]$	$U[\mathrm{m/s}]$	R_m	$\tau_{\rm d}$
earth (outer core)	2	10^6	10^{-3}	300	10^4 years
sun (plasma conductivity)	1	10^8	100	10^{10}	10^9 years
sun (turbulent conductivity)	10^8	10^8	100	100	3 years
liquid sodium lab experiment	0.1	1	10	100	10 s

Advection, diffusion, magnetic Reynolds number

$R_m \gg 1$ advection dominated regime (ideal MHD)

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B})$$

Equivalent expression

$$\frac{\partial \boldsymbol{B}}{\partial t} = -(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{B} + (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{v} - \boldsymbol{B}\,\boldsymbol{\nabla} \cdot \boldsymbol{v}$$

- advection of magnetic field
- amplification by shear (stretching of field lines)
- amplification through compression

Advection, diffusion, magnetic Reynolds number

Incompressible fluid ($\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$):

$$\frac{d\boldsymbol{B}}{dt} = (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{v}$$

Velocity shear in the direction of $\boldsymbol{B}$ plays key role. Mathematically similar equation for compressible fluid (Walen equation):

$$\frac{d}{dt}\frac{\boldsymbol{B}}{\varrho} = \left(\frac{\boldsymbol{B}}{\varrho} \cdot \boldsymbol{\nabla} \right) \boldsymbol{v}$$

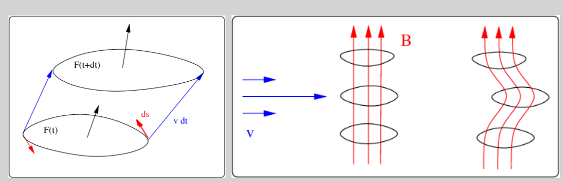
Vertical flux transport in statified medium:

- $B \sim \varrho$ no expansion in direction of $\boldsymbol{B}$
- $B \sim \varrho^{2/3}$ isotropic expansion
- $B \sim \varrho^{1/2}$ 2D expansion in plane containing $\boldsymbol{B}$
- $B = const.$ only expansion in direction of $\boldsymbol{B}$

Alfven's theorem

Let Φ be the magnetic flux through a surface F with the property that its boundary ∂F is moving with the fluid:

$$\Phi = \int_F \boldsymbol{B} \cdot d\boldsymbol{f} \longrightarrow \frac{d\Phi}{dt} = 0$$



- Flux is 'frozen' into the fluid
- Field lines 'move' with plasma

Dynamos: Motivation

- For $\boldsymbol{v} = 0$ magnetic field decays on timescale $\tau_d \sim L^2/\eta$
- Earth and other planets:
  - Evidence for magnetic field on earth for $3.5 \cdot 10^9$ years while $\tau_d \sim 10^4$ years
  - Permanent rock magnetism not possible since $T > T_{\rm Curie}$ and field highly variable $\longrightarrow$ field must be maintained by active process
- Sun and other stars:
  - Evidence for solar magnetic field for $\sim 300\,000$ years (^{10}Be)
  - Most solar-like stars show magnetic activity independent of age
  - Indirect evidence for stellar magnetic fields over life time of stars
  - But $\tau_d \sim 10^9$ years!
  - Primordial field could have survived in radiative interior of sun, but convection zone has much shorter diffusion time scale ~ 10 years (turbulent diffusivity)

Mathematical definition of dynamo

S bounded volume with the surface ∂S, $\boldsymbol{B}$ maintained by currents contained within S, $B \sim r^{-3}$ asymptotically,

$$\begin{aligned}
\frac{\partial \boldsymbol{B}}{\partial t} &= \nabla \times (\boldsymbol{v} \times \boldsymbol{B} - \eta \nabla \times \boldsymbol{B}) && \text{in } S \\
\nabla \times \boldsymbol{B} &= 0 && \text{outside } S \\
[\boldsymbol{B}] &= 0 && \text{across } \partial S \\
\nabla \cdot \boldsymbol{B} &= 0
\end{aligned}$$

$\boldsymbol{v} = 0$ outside S, $\boldsymbol{n} \cdot \boldsymbol{v} = 0$ on ∂S and

$$E_{\rm kin} = \int_S \frac{1}{2} \varrho \boldsymbol{v}^2 \, dV \le E_{\rm max} \quad \forall\, t$$

$\boldsymbol{v}$ is a dynamo if an initial condition $\boldsymbol{B} = \boldsymbol{B}_0$ exists so that

$$E_{\rm mag} = \int_{-\infty}^{\infty} \frac{1}{2\mu_0} \boldsymbol{B}^2 \, dV \ge E_{\rm min} \quad \forall\, t$$

Mathematical definition of dynamo

- Is this dynamo different from those found in powerplants?
  - Both have conducting material and relative motions (rotor/stator in powerplant vs. shear flows)
- Difference mostly in one detail:
  - Dynamos in powerplants have wires (very inhomogeneous conductivity), i.e. the electric currents are strictly controlled
  - Mathematically the system is formulated in terms of currents
  - A short circuit is a major desaster!
  - For astrophysical dynamos we consider homogeneous conductivity, i.e. current can flow anywhere
  - Mathematically the system is formulated in terms of $\boldsymbol{B}$ ($\boldsymbol{j}$ is eliminated from equations whenever possible).
  - A short circuit is the normal mode of operation!
- Homogeneous vs. inhomogeneous dynamos

Large scale/small scale dynamos

Decompose the magnetic field into large scale part and small scale part (energy carrying scale of turbulence) $\boldsymbol{B} = \overline{\boldsymbol{B}} + \boldsymbol{B}'$:

$$E_{\rm mag} = \int \frac{1}{2\mu_0} \overline{\boldsymbol{B}}^2 \, dV + \int \frac{1}{2\mu_0} \overline{\boldsymbol{B}'^2} \, dV \,.$$

- Small scale dynamo: $\overline{\boldsymbol{B}}^2 \ll \overline{\boldsymbol{B}'^2}$
- Large scale dynamo: $\overline{\boldsymbol{B}}^2 \ge \overline{\boldsymbol{B}'^2}$

Almost all turbulent (chaotic) velocity fields are small scale dynamos for sufficiently large R_m, large scale dynamos require additional large scale symmetries (see second half of this lecture)

What means large/small in practice (Sun)?

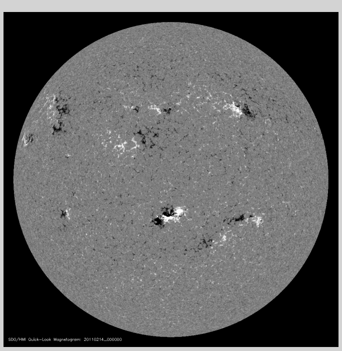


Figure: Full disk magnetogram SDO/HMI

What means large/small in practice (Sun)?

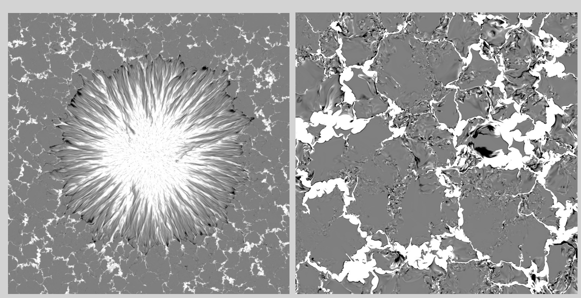


Figure: Numerical sunspot simulation. Dimensions: Left 50x50 Mm, Right: 12.5x12.5 Mm

Small scale dynamo action

Lagrangian particle paths:

$$\frac{d\boldsymbol{x}_1}{dt} = \boldsymbol{v}(\boldsymbol{x}_1, t) \qquad \frac{d\boldsymbol{x}_2}{dt} = \boldsymbol{v}(\boldsymbol{x}_2, t)$$

Consider small separations:

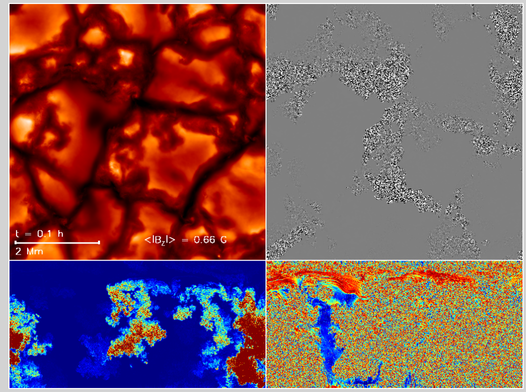
$$\boldsymbol{\delta} = \boldsymbol{x}_1 - \boldsymbol{x}_2 \qquad \frac{d\boldsymbol{\delta}}{dt} = (\boldsymbol{\delta} \cdot \boldsymbol{\nabla})\boldsymbol{v}$$

Chaotic flows have exponentially growing solutions. Due to mathematical simularity the equation:

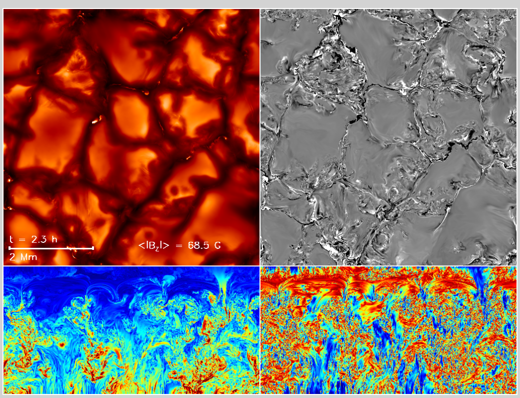
$$\frac{d}{dt}\frac{\boldsymbol{B}}{\varrho} = \left(\frac{\boldsymbol{B}}{\varrho} \cdot \boldsymbol{\nabla}\right)\boldsymbol{v}$$

has exponentially growing solutions, too. We neglected here η, exponentially growing solutions require $R_m > \mathcal{O}(100)$.

SSD in solar photosphere: kinematic phase

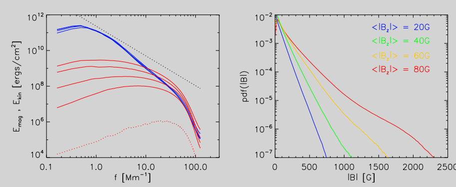


SSD in solar photosphere: saturated phase



Movie

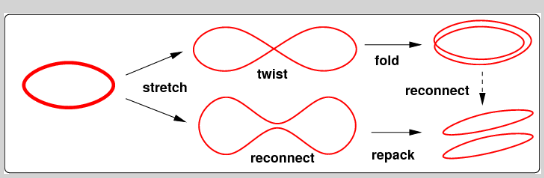
SSD in solar photosphere: power spectra



Movie

- Kinematic phase: Magnetic energy peaks at smallest resolved scales (here 30 km (4 km numerical resolution, would be 100 – 1000 m for the Sun
- Saturated phase: Magnetic energy peaks at granular scales (mostly flat spectrum at large scales). Dynamo action moved toward larger scales, where most of the kinetic energy sits (downflow lanes $\sim$ 300 km)

Large scale/small scale dynamos



- Amplification through field line stretching
- Twist-fold required to repack field into original volume
- Twist-fold requires 3D - there are no dynamos is 2D!
- Magnetic diffusivity allows for change of topology

Slow/fast dynamos

Influence of magnetic diffusivity on growth rate

- Fast dynamo: growth rate independent of R_m (stretch-twist-fold mechanism)
- Slow dynamo: growth rate limited by resistivity (stretch-reconnect-repack)

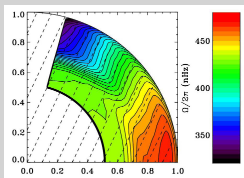
- Fast dynamos relevant for most astrophysical objects since $R_m \gg 1$
- Dynamos including (resistive) reconnection steps can be fast provided the reconnection is fast

Differential rotation and meridional flow

Induction effects of axisymmetric flows on axisymmetric field:

$$\boldsymbol{B} = B\boldsymbol{e_\Phi} + \boldsymbol{\nabla}\times(A\boldsymbol{e_\Phi})$$
$$\boldsymbol{v} = v_r\boldsymbol{e_r} + v_\theta\boldsymbol{e_\theta} + \Omega\, r\sin\theta\,\boldsymbol{e_\Phi}$$

Differential rotation most dominant shear flow in stellar convection zones:



Meridional flow by-product of DR, observed as poleward surface flow in case of the sun

Differential rotation and meridional flow

Spherical geometry:

$$\frac{\partial B}{\partial t} + \frac{1}{r}\left(\frac{\partial}{\partial r}(r v_r B) + \frac{\partial}{\partial \theta}(v_\theta B)\right) = r\sin \boldsymbol{B}_p\cdot\boldsymbol{\nabla}\Omega + \eta\left(\Delta - \frac{1}{(r\sin\theta)^2}\right)B$$

$$\frac{\partial A}{\partial t} + \frac{1}{r\sin\theta}\boldsymbol{v}_p\cdot\boldsymbol{\nabla}(r\sin\theta A) = \eta\left(\Delta - \frac{1}{(r\sin\theta)^2}\right)A$$

- Meridional flow: Independent advection of poloidal and toroidal field
- Differential rotation: Source for toroidal field (if poloidal field not zero)
- Diffusion: Sink for poloidal and toroidal field
- No term capable of maintaining poloidal field against Ohmic decay!

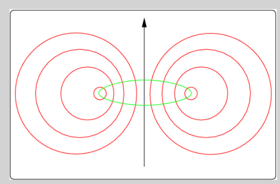
Differential rotation and meridional flow

- Weak poloidal seed field can lead to significant field amplification
- No source term for poloidal field
- Decay of poloidal field on resistive time scale
- Ultimate decay of toroidal field
- Not a dynamo!
- What is needed?

- Source for poloidal field

Cowling's anti-dynamo theorem

A stationary axisymmetric magnetic field with currents limited to a finite volume in space cannot be maintained by a velocity field with finite amplitude.



Ohm's law of the form $\boldsymbol{j} = \sigma\boldsymbol{E}$ only decaying solutions, focus here on $\boldsymbol{j} = \sigma(\boldsymbol{v}\times\boldsymbol{B})$.
On O-type neutral line $\boldsymbol{B}_p$ is zero, but $\mu_0\boldsymbol{j}_t = \boldsymbol{\nabla}\times\boldsymbol{B}_p$ has finite value, but cannot be maintained by $(\boldsymbol{v}\times\boldsymbol{B})_t = (\boldsymbol{v}_p\times\boldsymbol{B}_p)$.

Large scale dynamo theory

Some history:

- 1919 Sir Joeseph Larmor: Solar magnetic field maintained by motions of conducting fluid?
- 1937 Cowling's anti-dynamo theorem and many others
- 1955 Parker: decomposition of field in axisymmetric and non-axisymmetric parts, average over induction effects of non-axisymmetric field
- 1964 Braginskii, Steenbeck, Krause: Mathematical frame work of mean field theory developed
- last 2 decades 3D dynamo simulations

Reynolds rules

We need to define an averaging procedure to define the mean and the fluctuating field.
For any function f and g decomposed as $f = \overline{f} + f'$ and $g = \overline{g} + g'$ we require that the Reynolds rules apply

$$\overline{\overline{f}} = \overline{f} \longrightarrow \overline{f'} = 0$$
$$\overline{f+g} = \overline{f} + \overline{g}$$
$$\overline{\overline{f}g} = \overline{f}\,\overline{g} \longrightarrow \overline{f'\overline{g}} = 0$$
$$\overline{\partial f/\partial x_i} = \partial\overline{f}/\partial x_i$$
$$\overline{\partial f/\partial t} = \partial\overline{f}/\partial t\,.$$

Examples:

- Longitudinal average (mean = axisymmetric component)
- Ensemble average (mean = average over several realizations of chaotic system)

Meanfield induction equation

Average of induction equation:

$$\frac{\partial \overline{\boldsymbol{B}}}{\partial t} = \boldsymbol{\nabla} \times \left(\overline{\boldsymbol{v}' \times \boldsymbol{B}'} + \overline{\boldsymbol{v}} \times \overline{\boldsymbol{B}} - \eta \boldsymbol{\nabla} \times \overline{\boldsymbol{B}}\right)$$

New term resulting from small scale effects:

$$\overline{\boldsymbol{\mathcal{E}}} = \overline{\boldsymbol{v}' \times \boldsymbol{B}'}$$

Fluctuating part of induction equation:

$$\left(\frac{\partial}{\partial t} - \eta \Delta\right) \boldsymbol{B}' - \boldsymbol{\nabla} \times (\overline{\boldsymbol{v}} \times \boldsymbol{B}') = \boldsymbol{\nabla} \times \left(\boldsymbol{v}' \times \overline{\boldsymbol{B}} + \boldsymbol{v}' \times \boldsymbol{B}' - \overline{\boldsymbol{v}' \times \boldsymbol{B}'}\right)$$

Kinematic approach: $\boldsymbol{v}'$ assumed to be given

- Solve for $\boldsymbol{B}'$, compute $\overline{\boldsymbol{v}' \times \boldsymbol{B}'}$ and solve for $\overline{\boldsymbol{B}}$
- Term $\boldsymbol{v}' \times \boldsymbol{B}' - \overline{\boldsymbol{v}' \times \boldsymbol{B}'}$ leading to higher order correlations (closure problem)

Mean field expansion of turbulent induction effects

Exact expressions for $\overline{\boldsymbol{\mathcal{E}}}$ exist only under strong simplifying assumptions (see homework assignment).

In general $\overline{\boldsymbol{\mathcal{E}}}$ is a linear functional of $\overline{\boldsymbol{B}}$:

$$\overline{\mathcal{E}}_i(\boldsymbol{x}, t) = \int_{-\infty}^{\infty} d^3x' \int_{-\infty}^{t} dt' \, \mathcal{K}_{ij}(\boldsymbol{x}, t, \boldsymbol{x}', t') \, \overline{B}_j(\boldsymbol{x}', t') \, .$$

Can be simplified if a sufficient scale separation is present:

- $l_c \ll L$
- $\tau_c \ll \tau_L$

Leading terms of expansion:

$$\overline{\mathcal{E}}_i = a_{ij} \overline{B}_j + b_{ijk} \frac{\partial \overline{B}_j}{\partial x_k}$$

In stellar convection zones scale separation also only marginally justified (continuous turbulence spectrum)!

Symmetry constraints

Decomposing a_{ij} and $\partial \overline{B}_j / \partial x_k$ into symmetric and antisymmetric components:

$$a_{ij} = \underbrace{\frac{1}{2}(a_{ij} + a_{ji})}_{\alpha_{ij}} + \underbrace{\frac{1}{2}(a_{ij} - a_{ji})}_{-\varepsilon_{ijk}\gamma_k}$$

$$\frac{\partial \overline{B}_j}{\partial x_k} = \frac{1}{2}\left(\frac{\partial \overline{B}_j}{\partial x_k} + \frac{\partial \overline{B}_k}{\partial x_j}\right) + \underbrace{\frac{1}{2}\left(\frac{\partial \overline{B}_j}{\partial x_k} - \frac{\partial \overline{B}_k}{\partial x_j}\right)}_{-\frac{1}{2}\varepsilon_{jkl}(\boldsymbol{\nabla} \times \overline{\boldsymbol{B}})_l}$$

Leads to:

$$\overline{\mathcal{E}}_i = \alpha_{ij} \overline{B}_j + \varepsilon_{ikj} \gamma_k \overline{B}_j - \underbrace{\frac{1}{2} b_{ijk} \varepsilon_{jkl}}_{\beta_{il} - \varepsilon_{ilm}\delta_m} (\boldsymbol{\nabla} \times \overline{\boldsymbol{B}})_l + \dots$$

Symmetry constraints

Overall result:

$$\overline{\boldsymbol{\mathcal{E}}} = \boldsymbol{\alpha} \overline{\boldsymbol{B}} + \boldsymbol{\gamma} \times \overline{\boldsymbol{B}} - \boldsymbol{\beta} \, \boldsymbol{\nabla} \times \overline{\boldsymbol{B}} - \boldsymbol{\delta} \times (\boldsymbol{\nabla} \times \overline{\boldsymbol{B}}) + \dots$$

With:

$$\alpha_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) \, , \qquad \gamma_i = -\frac{1}{2} \varepsilon_{ijk} a_{jk}$$

$$\beta_{ij} = \frac{1}{4}(\varepsilon_{ikl} b_{jkl} + \varepsilon_{jkl} b_{ikl}) \, , \qquad \delta_i = \frac{1}{4}(b_{jji} - b_{jij})$$

Mean field induction equation

Induction equation for $\overline{\boldsymbol{B}}$:

$$\frac{\partial \overline{\boldsymbol{B}}}{\partial t} = \boldsymbol{\nabla} \times \left[\boldsymbol{\alpha} \overline{\boldsymbol{B}} + (\overline{\boldsymbol{v}} + \boldsymbol{\gamma}) \times \overline{\boldsymbol{B}} - (\eta + \boldsymbol{\beta}) \, \boldsymbol{\nabla} \times \overline{\boldsymbol{B}} - \boldsymbol{\delta} \times (\boldsymbol{\nabla} \times \overline{\boldsymbol{B}})\right]$$

Interpretation on first sight:

- $\boldsymbol{\alpha}$: new effect
- $\boldsymbol{\gamma}$: acts like advection (turbulent advection effect)
- $\boldsymbol{\beta}$: acts like diffusion (turbulent diffusivity)
- $\boldsymbol{\delta}$: special anisotropy of diffusion tensor