Instruments and observation techniques: astrophysical spectropolarimetry

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I. Propagation of light in media: polarization
II. Stokes vectors and Mueller matrices: telescope polarization
III. Modern spectropolarimeters

NOTE ON THE POLARIZING EFFECT OF COELOSTAT MIRRORS[1]

By CHARLES E. ST. JOHN

The discovery of the Zeeman effect in sun-spots by Hale[2] made it important to determine the action of the silver-on-glass mirrors of the tower telescope of the Mount Wilson Solar Observatory upon circularly polarized light. It was at once recognized that circularly polarized light would be changed to elliptically polarized by reflection from the silver surfaces and that its effect would vary with the angles of incidence and hence with the position of the sun. A description of the tower telescope has been given by Hale[3] and it only needs be said here that the second mirror sends the light vertically downward through the 12-inch objective and that about 4 feet below the objective—focal length 60 feet—the beam was received upon the analyzer which served to fix the position of the axes of the elliptically polarized light and to determine their ratio.

Handbook of Optics Chapter 12 on Polarization (J.M. Bennett)

which also represents an exponentially damped wave traveling in the $+z$ direction *provided that the complex index of refraction is defined to be*

$$\tilde{n}' = n + ik \tag{9}$$

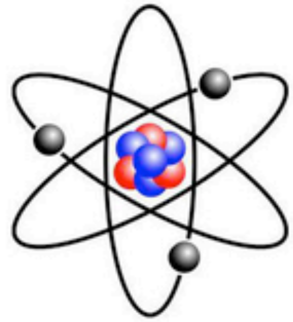
where the primes indicate the alternative solution. When the wave equation arises in quantum mechanics, the solution chosen is generally the negative exponential, i.e., Eq. (8) rather than Eq. (4). Solid-state physicists working in optics thus often define the complex index of refraction as the form given in Eq. (9) rather than that in Eq. (3). Equally valid, self-consistent theories can be built up using either definition, and as long as only intensities are considered, the resulting expressions are identical. However, when phase differences are calculated, the two conventions usually lead to contradictory results. Even worse, an author who is not extremely careful may not consistently follow either convention, and the result may be pure nonsense. Some well-known books might be cited in which the authors are not even consistent from chapter to chapter.

There are several other cases in optics in which alternative conventions are possible and both are found in the literature. Among these, the most distressing are the use of a left-handed rather than a right-handed coordinate system, which makes the p and s components of polarized light have the same phase change at normal incidence (see Sec. 12.3), and defining the optical constants so that they depend on the angle of incidence, which makes the angle of refraction given by Snell's law real for an absorbing medium. There are many advantages to be gained by using a single set of conventions in electromagnetic theory. In any event, an author should *clearly* state the conventions being used and then *stay with them*.

Furthermore, there are *sign errors that do cause physical differences* "sprinkled richly" throughout the literature on this subject. If you need a reference without sign errors, always consult the works of Landi degl' Innocenti! (B. Lites)

I. Propagation of light in media: polarization

1. Max Born & Emil Wolf: "Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light". 7th Edition
2. Landi Degl'Innocenti, Egidio, "The physics of polarization" Astrophysical spectropolarimetry. Proceedings of the XII Canary Islands Winter School of Astrophysics, edited by J. Trujillo-Bueno, F. Moreno-Insertis, and F. Sánchez. Cambridge, UK: Cambridge University Press, 2002, p. 1 – 53
3. Landi Degl'Innocenti, E. & Landolfi, M.: "Polarization in Spectral Lines", 2004. Kluwer Academic Publishers, Dordrecht
4. C.U. Keller http://home.strw.leidenuniv.nl/~keller/education.shtml
5. Jose Carlos del Toro Iniesta : "Introduction to Spectropolarimetry", 2003. Cambridge.
6. Lites, B. W. at IAA Granada: http://spg.iaa.es/pub/downloads/B.Lites.zip



KEEP CALM

AND USE

MAXWELL'S EQUATIONS

- **Maxwell Equations** (Gaussian units)

$$\bar{\nabla}\cdot\bar{D}=4\pi\rho$$

$$\bar{\nabla}\cdot\bar{B}=0$$

$$\bar{\nabla}\times\bar{E}+\frac{1}{c}\frac{\partial\bar{B}}{\partial t}=\bar{0}$$

$$\bar{\nabla}\times\bar{H}-\frac{1}{c}\frac{\partial\bar{D}}{\partial t}=\frac{4\pi}{c}\bar{J}$$

$\bar{E}$ = Electric Field

$\bar{H}$ = Magnetic Field

$\bar{B}$ = Magnetic Induction

$\bar{D}$ = Electric Displacement

$\bar{J}$ = Electric Current Density

ρ = Electric Charge Density

- Material equations

$$\bar{D}=\hat{\varepsilon}\bar{E}$$

$$\bar{J}=\hat{\sigma}\bar{E}$$

$$\bar{B}=\mu\bar{H}$$

$\hat{\varepsilon}$ = Dielectric tensor

$\hat{\sigma}$ = Specific Conductivity tensor

μ = Magnetic permeability

- In vacuum (≈air):

$$\hat{\sigma} = \hat{0} \quad \hat{\varepsilon} = \hat{1} \quad \mu = 1$$

$$\bar{D} = \bar{E} \quad \bar{B} = \bar{H} \quad \bar{J} = \bar{0} \quad \rho = 0$$

- Isotropic and anisotropic dielectrics (lots of optics):

$$\hat{\sigma} = 0\delta_{ij} \quad \left[\varepsilon\right]_{ij} = \varepsilon\delta_{ij} \quad \mu = 1$$

$$\hat{\sigma} = \hat{0} \quad \left[\varepsilon\right]_{ij} = \varepsilon_{ij} \quad \mu = 1$$

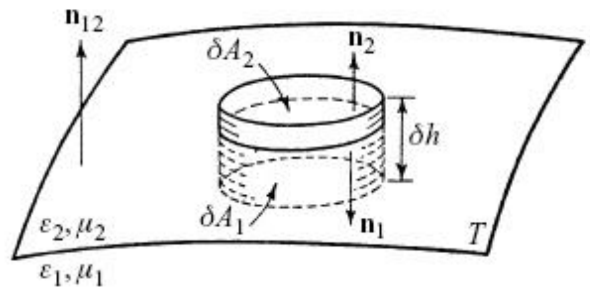
- Conductors (mirrors, stellar atmospheres):

$$\left[\sigma\right]_{ij} = \sigma\delta_{ij} \quad \left[\varepsilon\right]_{ij} = \varepsilon\delta_{ij} \quad \mu = 1 \qquad \bar{J} \neq \bar{0} \qquad \rho \neq 0$$

$$\left[\sigma\right]_{ij} = \sigma_{ij} \quad \left[\varepsilon\right]_{ij} = \varepsilon_{ij} \quad \mu = 1$$

- https://www.youtube.com/watch?v=wahmW7h-AKo
- Maxwell equations: **Boundary conditions**
- Differential equations → point. Area/Volume → integral
- Gauss and Stokes theorems of calculus

$$\int_V div\bar{D}\,dV = \oint_S \bar{D}\cdot\bar{n}\,dS = 4\pi\int_V \rho\,dV \xrightarrow[\delta h\to 0]{} \int_{\delta A}\hat{\rho}\,dA$$



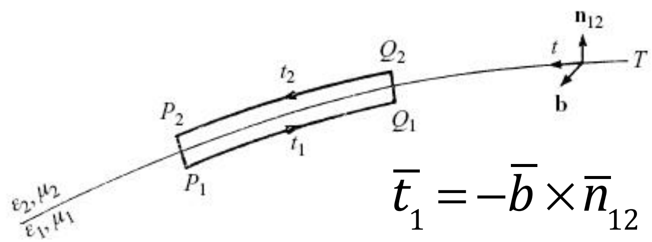
$$\boxed{\bar{n}_{12}\cdot(\bar{D}_2-\bar{D}_1)=4\pi\hat{\rho}(=0)}$$

$$\boxed{\bar{n}_{12}\cdot(\bar{B}_2-\bar{B}_1)=0}$$

$$\int_S \nabla\times\bar{H}\cdot\bar{b}\,dS = \oint_L \bar{H}\cdot d\bar{r} = \frac{1}{c}\int_S \frac{\partial\bar{D}}{\partial t}\cdot\bar{b}\,dS + \frac{4\pi}{c}\int_S \bar{J}\cdot\bar{b}\,dS \xrightarrow[\delta h(P_1P_2)\to 0]{} \frac{1}{c}\frac{\partial\bar{D}}{\partial t}\delta s\delta h + \frac{4\pi}{c}\int_{\delta s}\hat{\bar{J}}\cdot\bar{b}\,dr$$

$$\int_S \nabla\times\bar{E}\cdot\bar{b}\,dS = \oint_L \bar{E}\cdot d\bar{r} = -\frac{1}{c}\int_S \frac{\partial\bar{B}}{\partial t}\cdot\bar{b}\,dS \xrightarrow[\delta h(P_1P_2)\to 0]{} -\frac{1}{c}\frac{\partial\bar{B}}{\partial t}\delta s\delta h = 0$$

$$\boxed{\bar{n}_{12}\times(\bar{H}_2-\bar{H}_1)=\frac{4\pi}{c}\hat{\bar{J}}(=\bar{0})}$$



$$\boxed{\bar{n}_{12}\times(\bar{E}_2-\bar{E}_1)=0}$$

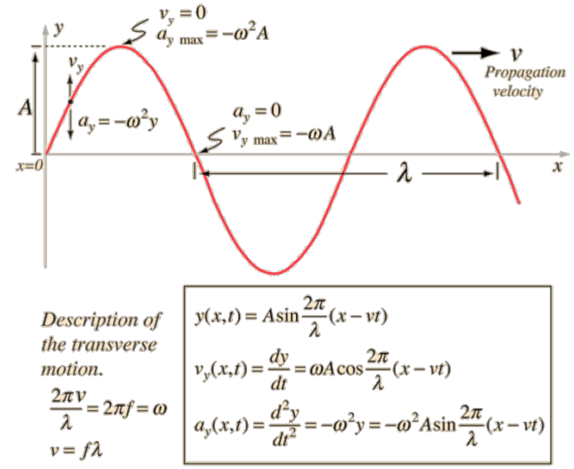
$$\bar{t}_1 = -\bar{b}\times\bar{n}_{12}$$

$$\bar{t}_2 = \bar{b}\times\bar{n}_{12}$$

- In vacuum one obtains the wave equation (V one component):

$$\nabla^2 \overline{E} - \frac{\varepsilon\mu}{c^2}\frac{\partial^2 \overline{E}}{\partial t^2} = \overline{0} \qquad \nabla^2 V - \frac{1}{v^2}\frac{\partial^2 V}{\partial t^2} = 0$$

$$v = \frac{c}{\sqrt{\varepsilon\mu}} \qquad n = \frac{c}{v} = \sqrt{\varepsilon\mu} \approx \sqrt{\varepsilon}$$



- Plane (propagating in direction $\overline{s}$) and spherical wave:

$$V(\overline{r}\cdot\overline{s} \pm vt) \qquad \overline{r}\cdot\overline{s} = cte$$

$$\frac{1}{r}V(r \pm vt) \qquad r = cte$$

- EM plane waves are transverse:

$$\overline{E} = \overline{E}(\overline{r}\cdot\overline{s} - vt) = \overline{E}(u) \qquad u = \overline{r}\cdot\overline{s} - vt$$

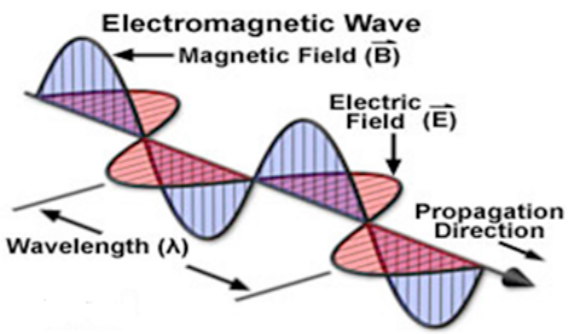
$$\overline{H} = \overline{H}(\overline{r}\cdot\overline{s} - vt) = \overline{H}(u)$$

- Maxwell Equation tell us that EM waves are transverse:

$$\bar{\nabla}\times\bar{E}+\frac{1}{c}\frac{\partial\bar{B}}{\partial t}=\bar{0} \qquad \bar{\nabla}\times\bar{B}-\frac{\varepsilon}{c}\frac{\partial\bar{E}}{\partial t}=\bar{0}$$

$$\left[\nabla\times\bar{E}\right]_x=(\bar{s}\times\frac{d\bar{E}}{du})_x \qquad \frac{\partial\bar{E}}{\partial t}=-v\frac{d\bar{E}}{du}$$

$$\bar{s}\times\frac{d\bar{E}}{du}-\frac{v}{c}\frac{d\bar{B}}{du}=\bar{0}$$



$$\bar{s}\times\frac{d\bar{B}}{du}+\frac{v\varepsilon}{c}\frac{d\bar{E}}{du}=\bar{0}$$

$$\bar{E}=-\frac{1}{\sqrt{\varepsilon}}\left(\bar{s}\times\bar{B}\right) \qquad \bar{B}=\sqrt{\varepsilon}\left(\bar{s}\times\bar{E}\right)$$

$$\boxed{\bar{S}=\frac{c}{4\pi}(\bar{E}\times\bar{B})\sim\bar{s}} \qquad \left|\bar{S}\right|=\frac{cn}{4\pi}E^2$$

- Monochromatic (single frequency) plane wave:

$$\bar{k} = \frac{2\pi}{\lambda}\bar{s} \qquad \varpi = 2\pi \nu \qquad v = \lambda \cdot \nu \qquad n = \frac{c}{v}$$

$$u = \bar{r}\cdot\bar{s} - vt = \frac{\lambda}{2\pi}(\bar{k}\cdot\bar{r} - \varpi t) = v\left[\frac{n}{c}\bar{r}\cdot\bar{s} - t\right] = \frac{\lambda\varpi}{2\pi}\left(\frac{\bar{r}\cdot\bar{s}}{v} - t\right)$$

- Complex notation, phases (note that we use $e^{-i\varpi t + i\delta}$):

$$\bar{E} = \bar{E}_o \cos(\bar{k}\cdot\bar{r} - \varpi t) = \mathrm{Re}\left\{\bar{E}_o e^{i(\bar{k}\cdot\bar{r} - \varpi t)}\right\}$$

$$E_x = \mathrm{Re}\left\{\left(E_o\right)_x e^{i(\tau + \delta_x)}\right\}$$

$$E_y = \mathrm{Re}\left\{\left(E_o\right)_y e^{i(\tau + \delta_y)}\right\}$$

$$E_z = 0$$

$$\boxed{\tau = (\bar{k}\cdot\bar{r} - \varpi t)}$$

- **Polarization ellipse** at a point in space (BW uses + for phases)

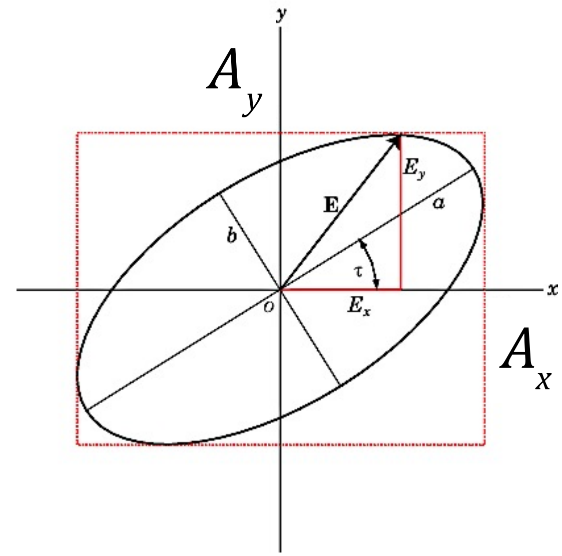
$$E_x = \mathrm{Re}\left\{ \left(E_o\right)_x e^{i(\tau+\delta_x)} \right\} = A_x \cos(\varpi t - \delta_x)$$

$$E_y = \mathrm{Re}\left\{ \left(E_o\right)_y e^{i(\tau+\delta_y)} \right\} = A_y \cos(\varpi t - \delta_y)$$

- Solving for ϖt

$$\delta = \delta_x - \delta_y$$

$$\left(\frac{E_x}{A_x}\right)^2 + \left(\frac{E_y}{A_y}\right)^2 - 2\frac{E_x}{A_x}\frac{E_y}{A_y}\cos\delta = \sin^2\delta$$



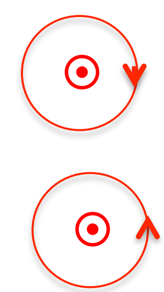
- Special cases: linear & circular

$$\delta = m\pi$$

$$m = 0, \pm 1, \pm 2, \ldots$$

$$\delta = m\frac{\pi}{2} \qquad A_x = A_y$$

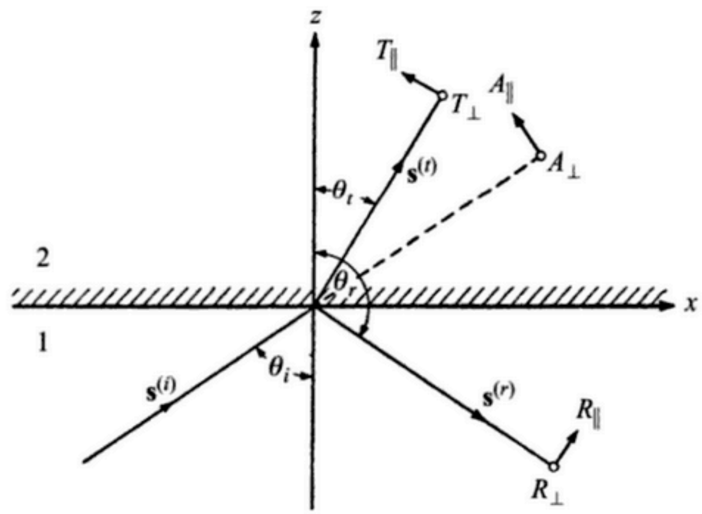
$$m = \pm 1, \pm 3, \ldots$$



$m = +1$

$m = -1$

- <u>Refraction and reflection in **dielectric media**</u>



$$\tau_i = (\bar{k}_i \cdot \bar{r} - \varpi_i t)$$

$$\tau_r = (\bar{k}_r \cdot \bar{r} - \varpi_r t)$$

$$\tau_t = (\bar{k}_t \cdot \bar{r} - \varpi_t t)$$

$$\tilde{A}_\parallel = A_\parallel e^{i\tau_i} \qquad \tilde{T}_\parallel = T_\parallel e^{i\tau_t} \qquad \tilde{R}_\parallel = R_\parallel e^{i\tau_r}$$

$$\tilde{A}_\perp = A_\perp e^{i(\tau_i+\delta_i)} \qquad \tilde{T}_\perp = T_\perp e^{i(\tau_t+\delta_t)} \qquad \tilde{R}_\perp = R_\perp e^{i(\tau_r+\delta_r)}$$

- Boundary conditions at interface → refraction/reflection laws:

$$(\bar{D}_2 - \bar{D}_1)\cdot\bar{n} = 0 \;;\; (\bar{B}_2 - \bar{B}_1)\cdot\bar{n} = 0 \;;\; \bar{n}\times(\bar{E}_2 - \bar{E}_1) = \bar{0} \;;\; \bar{n}\times(\bar{H}_2 - \bar{H}_1) = \bar{0}$$

$$\varpi_i = \varpi_r = \varpi_t \qquad \delta_i = \delta_t \neq \delta_r \qquad \boxed{\theta_i = \pi - \theta_r} \qquad \boxed{n_1 \sin\theta_i = n_2 \sin\theta_t}$$

- Fresnel formulae and Brewster angle:

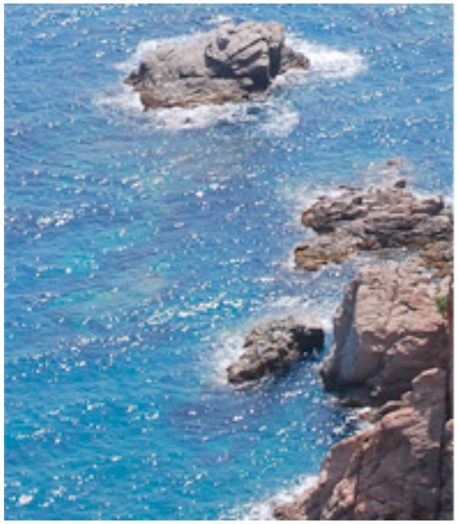
$$T_{\parallel} = \frac{2\sin\theta_t \cos\theta_i}{\sin(\theta_i + \theta_t)\cos(\theta_i - \theta_t)} A_{\parallel}$$

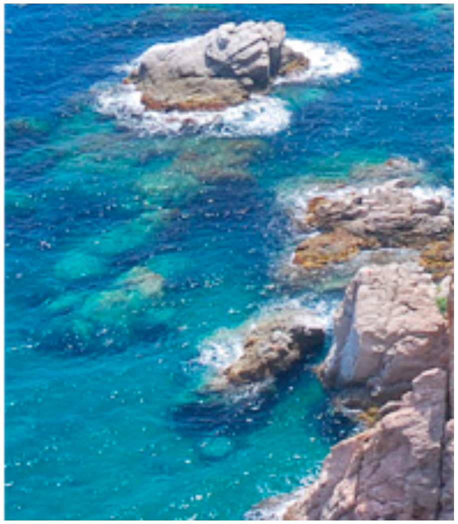
$$\theta_i + \theta_t = \frac{\pi}{2} \rightarrow R_{\parallel} = 0 \qquad \boxed{\tan\theta_i^B = \frac{n_2}{n_1}} \qquad \theta_i^B = 53^o$$

$$T_{\perp} = \frac{2\sin\theta_t \cos\theta_i}{\sin(\theta_i + \theta_t)} A_{\perp}$$

$$R_{\parallel} = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} A_{\parallel}$$

$$R_{\perp} = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)} A_{\perp}$$





- Total reflection

$$\sin\theta_t = \frac{n_1}{n_2}\sin\theta_i \geq 1$$

$$n_1(water) = 1.33$$

$$n_2(air) = 1$$

$$\theta_i \geq 49^o$$

- Amplitude ratios and phase difference

$$E_{\parallel}^{after} = P_{\parallel} e^{i\Delta_{\parallel}} E_{\parallel}^{before} \qquad E_{\perp}^{after} = P_{\perp} e^{i\Delta_{\perp}} E_{\perp}^{before}$$

$$\boxed{\frac{E_{\perp}^{after}}{E_{\parallel}^{after}} = P e^{i\Delta} \frac{E_{\perp}^{before}}{E_{\parallel}^{before}}}$$

$$P = \frac{P_{\perp}}{P_{\parallel}}$$

$$\Delta = \Delta_{\perp} - \Delta_{\parallel}$$

- Refraction and reflection in dielectric media

$$\frac{T_{\perp}}{T_{\parallel}} = \cos(\theta_i - \theta_t) \frac{A_{\perp}}{A_{\parallel}}$$

$$P = \cos(\theta_i - \theta_t)$$

$$\Delta = 0$$

$$\frac{R_{\perp}}{R_{\parallel}} = -\frac{\cos(\theta_i - \theta_t)}{\cos(\theta_i + \theta_t)} \frac{A_{\perp}}{A_{\parallel}}$$

$$P = \left| \frac{\cos(\theta_i - \theta_t)}{\cos(\theta_i + \theta_t)} \right|$$

$$\Delta = \pi \rightarrow (\theta_i + \theta_t) < \frac{\pi}{2}$$

$$\Delta = 0 \rightarrow (\theta_i + \theta_t) > \frac{\pi}{2}$$

- Total reflection $\theta_i \geq \sin^{-1}\left(\frac{n_2}{n_1}\right)$ $n_2 < n_1$

$$\cos\theta_t = \pm i\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_i - 1}$$

$$e^{i\tau_t} = e^{i(\bar{k}_t\cdot\bar{r} - \varpi t)} = e^{i\varpi\left(\frac{x n_1 \sin\theta_i}{n_2 v_2} - t\right)} e^{\mp\frac{\omega z}{v_2}\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_i - 1}}$$

- No transmitted wave: energy reflected

$$|R_\parallel| = |A_\parallel| \qquad |R_\perp| = |A_\perp|$$

$$\Delta = \Delta_\perp - \Delta_\parallel \qquad \frac{R_\perp}{R_\parallel} = e^{i\Delta}\frac{A_\perp}{A_\parallel} \qquad \tan\frac{\Delta}{2} = \frac{\cos\theta_i\sqrt{\sin^2\theta_i - \frac{n_2^2}{n_1^2}}}{\sin^2\theta_i}$$

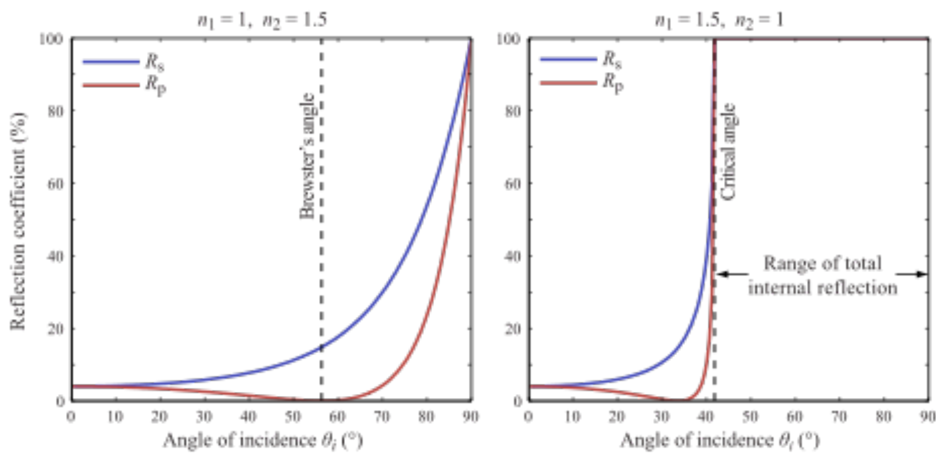
$$P = 1$$

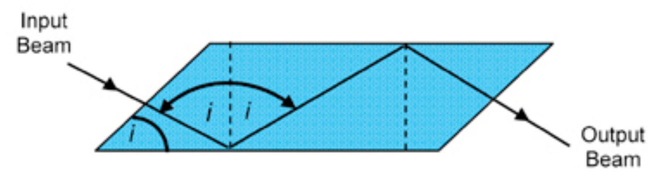
- Total reflection optical glasses

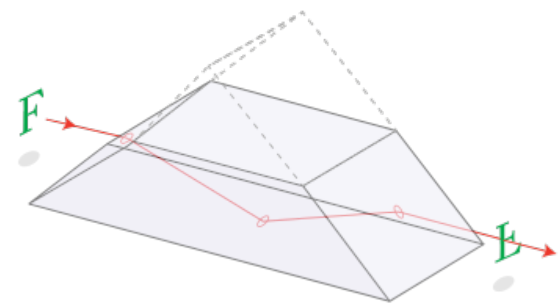
$n_1(BK7) = 1.5$

$n_2(air) = 1$

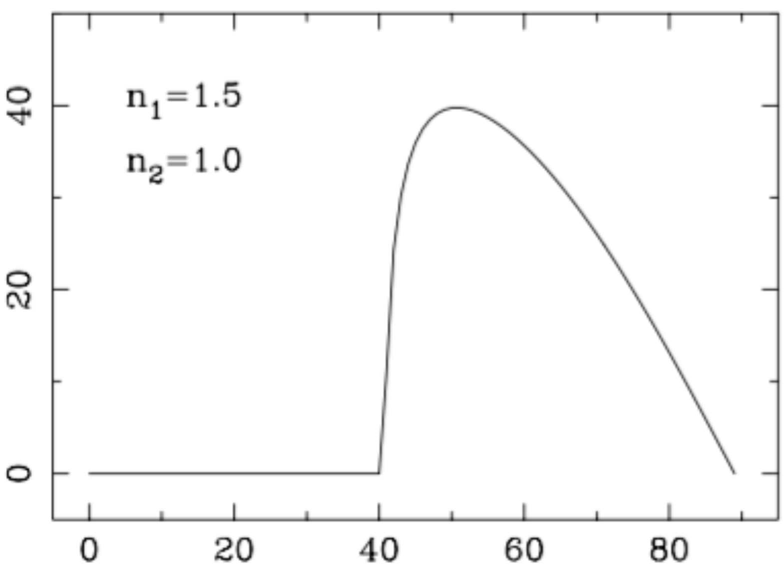
$\theta_i \geq 42^o$



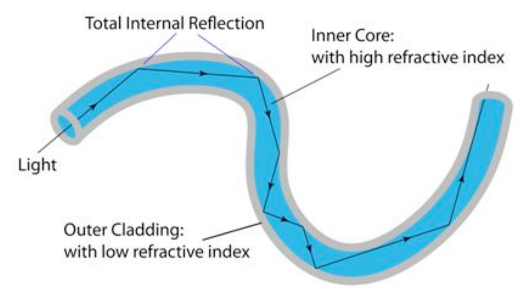




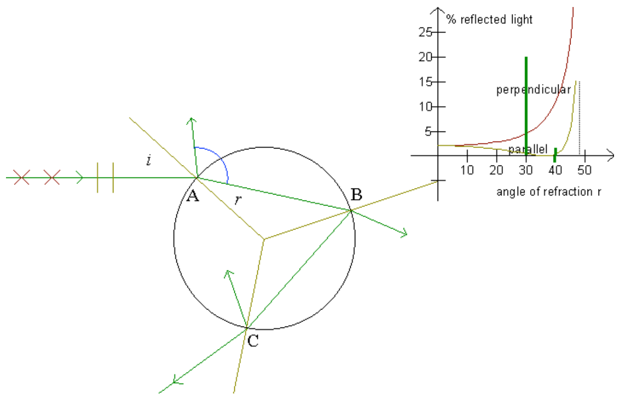
$\Delta(^o)$

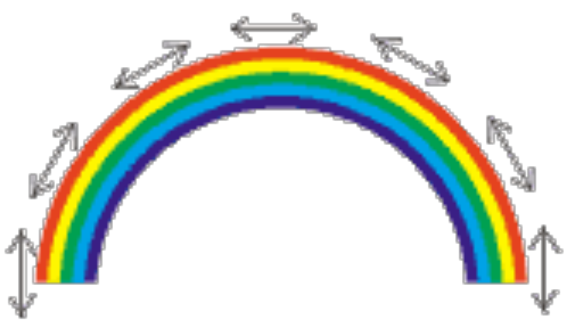


$\theta_i(^o)$



- The rainbow (2 refractions, 1 total reflection)





A polarized rainbow, what does this mean???

$$P_{rainbow} = \frac{(2+n^2)^6 - 729n^4(2-n^2)^2}{(2+n^2)^6 + 729n^4(2-n^2)^2}$$

- **Conducting media** (optics of metals):

$$\left[\sigma\right]_{ij}=\sigma\delta_{ij} \quad \left[\varepsilon\right]_{ij}=\varepsilon\delta_{ij} \quad \mu=1 \qquad \bar{J}=\sigma\bar{E} \qquad \rho=0$$

$$\bar{\nabla}\times\bar{H}-\frac{\varepsilon}{c}\frac{\partial\bar{E}}{\partial t}=\frac{4\pi}{c}\sigma\bar{E} \quad \longrightarrow \quad \nabla^2\bar{E}-\frac{\varepsilon}{c^2}\frac{\partial^2\bar{E}}{\partial t^2}-\frac{4\pi\sigma}{c^2}\frac{\partial\bar{E}}{\partial t}=0$$

- Damping term in the wave equation: $\bar{E}\propto e^{-i\varpi t}$

$$\nabla^2\bar{E}-\frac{\varpi^2\tilde{\varepsilon}}{c^2}\bar{E}=0 \qquad \tilde{\varepsilon}=\varepsilon+i\frac{4\pi\sigma}{\varpi} \qquad \tilde{n}=\frac{c}{\tilde{v}}=\sqrt{\tilde{\varepsilon}}=n(1+i\kappa)$$

$$\bar{E}=\bar{E}_o\cos\left(\tilde{k}(\bar{s}\cdot\bar{r})-\varpi t\right)=\mathrm{Re}\left\{\bar{E}_o e^{i\left(\tilde{k}(\bar{s}\cdot\bar{r})-\varpi t\right)}\right\} \qquad \tilde{k}=\frac{\varpi\tilde{n}}{c}$$

- κ Extinction or absorption coefficient (not unrelated to what you have seen !)

- **<u>Conducting media</u>** (optics of metals):

$$n^2 = \frac{1}{2}\left(\varepsilon + \sqrt{\varepsilon^2 + \frac{4\sigma^2}{\nu^2}}\right) \qquad n^2\kappa^2 = \frac{1}{2}\left(-\varepsilon + \sqrt{\varepsilon^2 + \frac{4\sigma^2}{\nu^2}}\right)$$

- The wave in the metallic media gets damped:

$$\bar{E} = \bar{E}_o e^{i(\tilde{k}(\bar{s}\cdot\bar{r}) - \varpi t)} = \bar{E}_o e^{-i\varpi t} e^{i\frac{\varpi n}{c}(\bar{s}\cdot\bar{r})} e^{-\frac{\varpi n\kappa}{c}(\bar{s}\cdot\bar{r})}$$

- The transmitted wave: $1 \cdot \sin\theta_i = \tilde{n}\sin\theta_t$

$$\sin\theta_t = \frac{1 - i\kappa}{n(1+\kappa^2)}\sin\theta_i$$

$$\cos\theta_t = \sqrt{1 - \frac{(1-\kappa^2)}{n^2(1+\kappa^2)^2}\sin^2\theta_i - i\frac{2\kappa}{n^2(1+\kappa^2)^2}\sin^2\theta_i}$$

- Fresnel equation for conducting media:

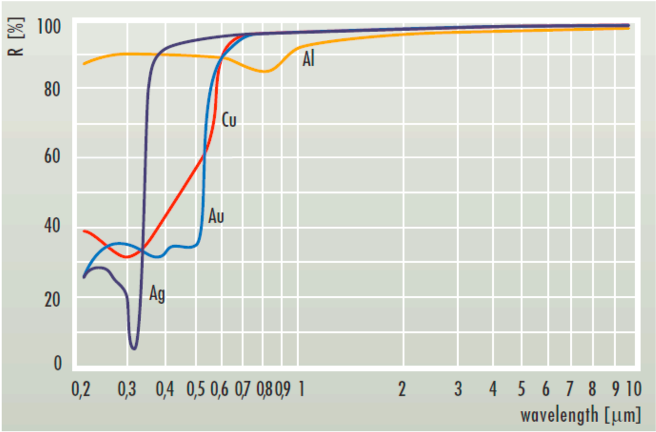
$$\frac{R_\perp}{R_\parallel} = -\frac{\cos(\theta_i - \theta_t)}{\cos(\theta_i + \theta_t)} \frac{A_\perp}{A_\parallel}$$

$$\frac{R_\perp}{R_\parallel} = P e^{i\Delta} \frac{A_\perp}{A_\parallel}$$

$$\theta_t = \theta_t(n, \kappa, \theta_i)$$

$$P = P(n, \kappa, \theta_i)$$

$$\Delta = \Delta(n, \kappa, \theta_i)$$



- Amplitude ratio and phases for reflection in metals:

$$P^2 = \frac{f^2 + g^2 + 2f\sin\theta_i\tan\theta_i + \sin^2\theta_i\tan^2\theta_i}{f^2 + g^2 - 2f\sin\theta_i\tan\theta_i + \sin^2\theta_i\tan^2\theta_i}$$

$$\tan\Delta = \frac{2g\sin\theta_i\tan\theta_i}{\sin^2\theta_i\tan^2\theta_i - (f^2 + g^2)}$$

$$f^2 = \frac{1}{2}\left[n^2 - n^2\kappa^2 - \sin^2\theta_i + \sqrt{(n^2 - n^2\kappa^2 - \sin^2\theta_i)^2 + 4n^4\kappa^2}\right]$$

$$g^2 = \frac{1}{2}\left[n^2\kappa^2 - n^2 + \sin^2\theta_i + \sqrt{(n^2 - n^2\kappa^2 - \sin^2\theta_i)^2 + 4n^4\kappa^2}\right]$$

- Normal incidence

$$\theta_i = 0 \longrightarrow \quad f^2 = n^2 \quad g^2 = n^2\kappa^2 \quad P^2 = 1 \quad \tan\Delta = 0$$

$$\frac{R_\perp}{R_\parallel} = -\frac{\cos(\theta_i - \theta_t)}{\cos(\theta_i + \theta_t)}\frac{A_\perp}{A_\parallel} = -\frac{A_\perp}{A_\parallel} \;\rightarrow\; \Delta = \pi$$

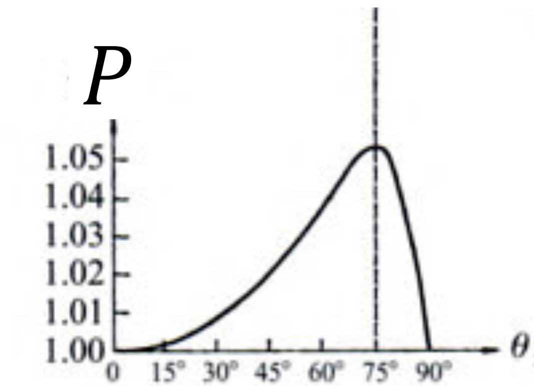
- Grazing incidence:

$$\theta_i = \frac{\pi}{2} \longrightarrow \quad \tan\theta_i \to \infty \quad P^2 = 1 \quad \tan\Delta = 0$$

$$\frac{R_\perp}{R_\parallel} = \frac{\sin\theta_t}{\sin\theta_t}\frac{A_\perp}{A_\parallel} = \frac{A_\perp}{A_\parallel} \;\rightarrow\; \Delta = 0$$

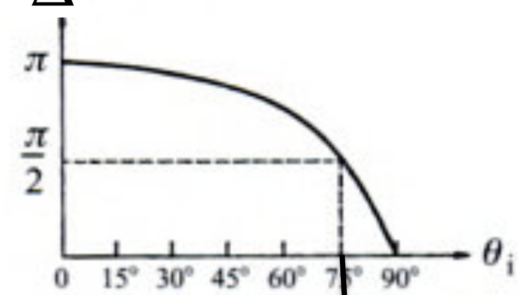
- General shape (example aluminum, 630nm):

P



$$f^2 + g^2 = \sin^2\theta_i \tan^2\theta_i$$

Δ



$$\tan\Delta = \infty$$

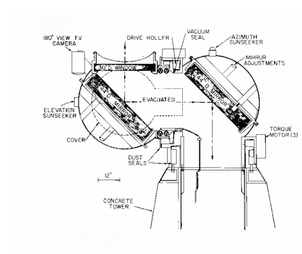
$n = 1.26$

$n\kappa = 7.25$

$\theta_i = 45^o$

$f =$ 1.2542; g= 7.7834

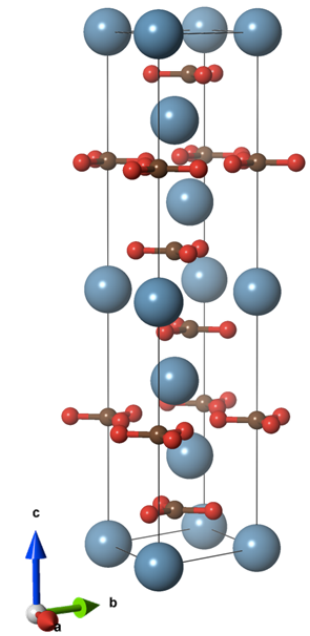
P=1.0327 ; $\Delta = 169.22^o$



http://refractiveindex.info/

- **<u>Anisotropic media</u>** <u>(crystals)</u>:

$$[\varepsilon]_{ij} = \varepsilon_{ij} \quad \sigma = 0 \quad \mu = 1 \qquad D_i = \varepsilon_{ij} E_j$$



- Dielectric tensor is diagonal in a reference frame:

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}$$

$$n_x = \sqrt{\varepsilon_x} \qquad n_y = \sqrt{\varepsilon_y} \qquad n_z = \sqrt{\varepsilon_z}$$

$$D_x = \varepsilon_x E_x \qquad D_y = \varepsilon_y E_y \qquad D_z = \varepsilon_z E_z$$

$$v_x = \frac{c}{\sqrt{\varepsilon_x}} \qquad v_y = \frac{c}{\sqrt{\varepsilon_y}} \qquad v_z = \frac{c}{\sqrt{\varepsilon_z}}$$

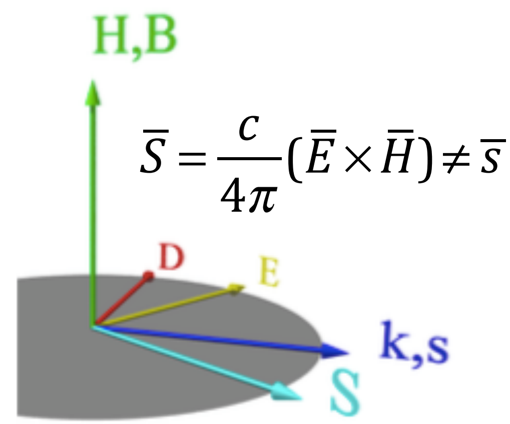
- In general $\bar{D}$ and $\bar{E}$ are not parallel.

$$\bar{D} = \bar{D}_o e^{i(\bar{k}\cdot\bar{r} - \varpi t)} \qquad \bar{\nabla}\cdot\bar{D} = 4\pi\rho = 0 \rightarrow \bar{D}\cdot\bar{s} = 0$$

$$\bar{E} = \bar{E}_o e^{i(\bar{k}\cdot\bar{r} - \varpi t)} \qquad \bar{\nabla}\cdot\bar{H} = 0 \rightarrow \bar{H}\cdot\bar{s} = 0$$

$$\bar{H} = \bar{H}_o e^{i(\bar{k}\cdot\bar{r} - \varpi t)} \qquad n\bar{s}\times\bar{H} = -\bar{D} \quad n\bar{s}\times\bar{E} = \bar{H}$$

$$\bar{S} = \frac{c}{4\pi}(\bar{E}\times\bar{H}) \neq \bar{s}$$



- **<u>Anisotropic media</u>** (crystals):

$$\bar{D} = n^2\left[\bar{E} - \bar{s}(\bar{E}\cdot\bar{s})\right] \quad \varepsilon_i E_i = n^2\left[E_i - s_i(\bar{E}\cdot\bar{s})\right] \quad E_i = \frac{n^2 s_i(\bar{E}\cdot\bar{s})}{(n^2-\varepsilon_i)}$$

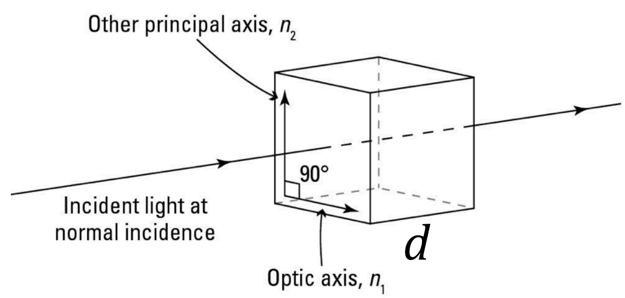
$$\frac{s_x^2}{n^2-\varepsilon_x} + \frac{s_y^2}{n^2-\varepsilon_y} + \frac{s_z^2}{n^2-\varepsilon_z} = \frac{1}{n^2}$$

- For each $\bar{s}$ quadratic equation for $n \rightarrow n_1, n_2$
- Double refraction: E_i has no phase→ linearly polarized
- $\bar{E}_1 \cdot \bar{E}_2 = 0$ polarized in orthogonal directions
- Uniaxial materials: $n_x = n_y \neq n_z \quad n_x = n_y = n_o;\ n_z = n_e$
- Optic axis is the axis that has a different index of refraction (z)
- $\bar{E}_o$ is $\perp$ to the to the principal plane $\bar{s}$ & optic axis. $\bar{E}_e$ is $\|$
- θ angle between $\bar{s}$ and the optic axis $s_x^2 + s_y^2 = \sin^2\theta \quad s_z^2 = \cos^2\theta$

$$(n_o^2 - n^2)\left[n_o^2\left(n_e^2 - n^2\right)\sin^2\theta + n_e^2\left(n_o^2 - n^2\right)\cos^2\theta\right] = 0$$

$$n_1 = n_o;\ n_2 = \frac{n_o n_e}{\sqrt{n_o^2\sin^2\theta + n_e^2\cos^2\theta}}$$

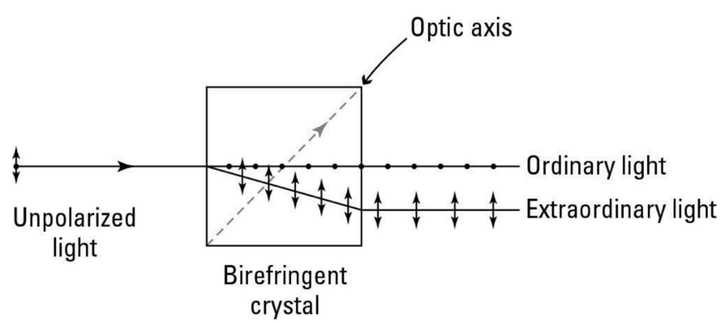
- **<u>Anisotropic media</u>** (crystals):
- $\theta = 0$ propagation along the optic axis $n_2 = n_o$ → isotropic
- $\theta = \frac{\pi}{2}$ propagation perpendicular to optic axis $n_2 = n_e$
- If $\theta_i = 0$ one ray sees n_o and the other n_e
- Phase delay between rays: birefringence $\delta = (n_e - n_o)$



$$\frac{R_\perp}{R_\parallel} = Pe^{i\Delta}\frac{A_\perp}{A_\parallel} \qquad \Delta = \frac{2\pi d}{\lambda}(n_o - n_e) = -\frac{2\pi d}{\lambda}\delta$$

$$P = 1$$

- $\perp$ and $\parallel$ refer to the principal plane ($\bar{s}$ and optic axis):
- At an arbitrary θ : two exit rays polarized $\perp$ directions



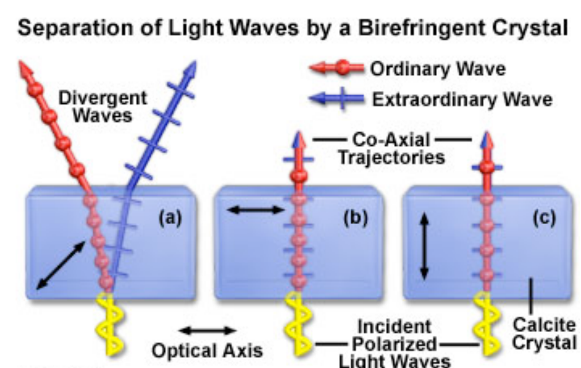


Figure 1

$\theta = 0$

- Quarter and half wave plates

$$\Delta = \frac{2\pi d}{\lambda}(n_o - n_e) = \frac{\pi}{2} \rightarrow d = \frac{\lambda}{4(n_o - n_e)}$$

$$\Delta = \frac{2\pi d}{\lambda}(n_o - n_e) = \pi \rightarrow d = \frac{\lambda}{2(n_o - n_e)}$$

$$\Delta = \frac{2\pi d}{\lambda}(n_o - n_e) = 2\pi \rightarrow d = \frac{\lambda}{(n_o - n_e)}$$

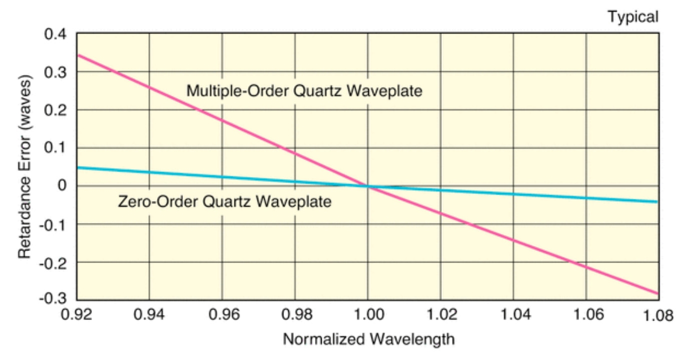
- Quartz: 14 μm, 28 μm: difficult
- Multi-order retarders:

$$d = \frac{N\lambda}{(n_o - n_e)}; \quad N = m + \frac{1}{4}$$

- Less achromatic than true zero order
- Compound zero order:
  - Two wave plates retardations differ by exactly λ/4 or λ/2
  - The fast axis of one plate aligned with the slow axis of the other
  - Net retardation is the difference of the two retardations
- Polymer retarders offer much better field of view

Uniaxial crystals, at 590 nm[5]

Material	Crystal system	n_o	n_e	Δn
barium borate BaB_2O_4	Trigonal	1.6776	1.5534	-0.1242
beryl $Be_3Al_2(SiO_3)_6$	Hexagonal	1.602	1.557	-0.045
calcite $CaCO_3$	Trigonal	1.658	1.486	-0.172
ice H_2O	Hexagonal	1.309	1.313	+0.004
lithium niobate $LiNbO_3$	Trigonal	2.272	2.187	-0.085
magnesium fluoride MgF_2	Tetragonal	1.380	1.385	+0.006
quartz SiO_2	Trigonal	1.544	1.553	+0.009
ruby Al_2O_3	Trigonal	1.770	1.762	-0.008
rutile TiO_2	Tetragonal	2.616	2.903	+0.287
sapphire Al_2O_3	Trigonal	1.768	1.760	-0.008
silicon carbide SiC	Hexagonal	2.647	2.693	+0.046
tourmaline (complex silicate)	Trigonal	1.669	1.638	-0.031
zircon, high $ZrSiO_4$	Tetragonal	1.960	2.015	+0.055



- **<u>Anisotropic conducting media</u>** <u>(metal crystals)</u>:

$$\left[\varepsilon\right]_{ij}=\varepsilon_{ij}\quad \left[\sigma\right]_{ij}=\sigma_{ij}\quad \mu=1\quad D_i=\varepsilon_{ij}E_j\quad J_i=\sigma_{ij}E_j$$

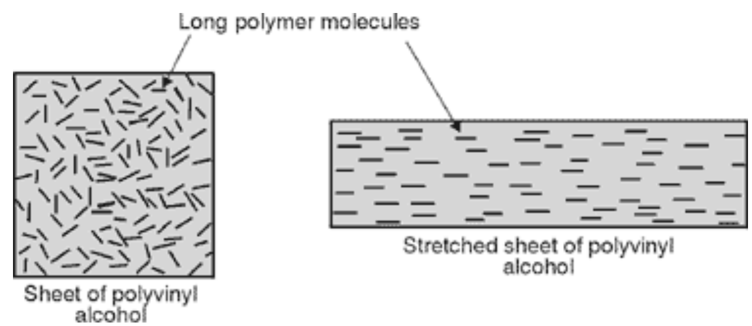
$$\tilde{\varepsilon}_i=\varepsilon_i+i\frac{4\pi\sigma_i}{\varpi}\quad i=x,y,z$$

- Each $\bar{s}$ has two complex refractive indexes

$$\tilde{n}_i=\frac{c}{\hat{v}_i}=\sqrt{\tilde{\varepsilon}_i}=n_i(1+i\kappa_i)\qquad \bar{E}_i=\bar{E}_{oi}e^{-i\varpi t}e^{i\frac{\varpi n_i}{c}(\bar{s}\cdot\bar{r})}e^{-\frac{\varpi n_i\kappa_i}{c}(\bar{s}\cdot\bar{r})}$$

- Dichroism (Polaroids):

$$\frac{R_\perp}{R_\parallel}=P\frac{A_\perp}{A_\parallel}\qquad P=e^{-\frac{\varpi}{c}nd(\kappa_\perp-\kappa_\parallel)}$$



- Ideal polarizer has

$$\kappa_\perp=\infty\quad(\kappa_\parallel=0)\quad\rightarrow\quad P=0$$

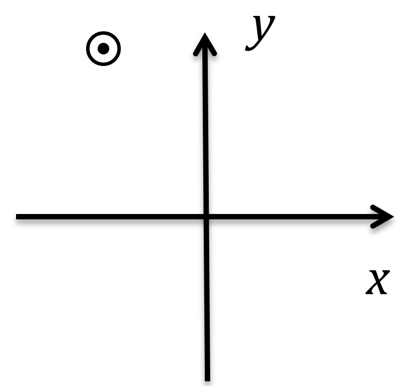
II. Stokes vectors and Mueller matrices: telescope polarization

- <u>Monochromatic plane wave: Jones vector</u>

$$E_x = A_x e^{i\delta_x} e^{i\tau}$$
$$E_y = A_y e^{i\delta_y} e^{i\tau}$$
$$\tau = (\bar{k}\cdot\bar{r} - \varpi t)$$

$$\bar{J} = \begin{pmatrix} A_x e^{i\delta_x} \\ A_y e^{i\delta_y} \end{pmatrix} = e^{i\delta_x} \begin{pmatrix} A_x \\ A_y e^{-i\delta} \end{pmatrix}$$

$$\delta = \delta_x - \delta_y$$

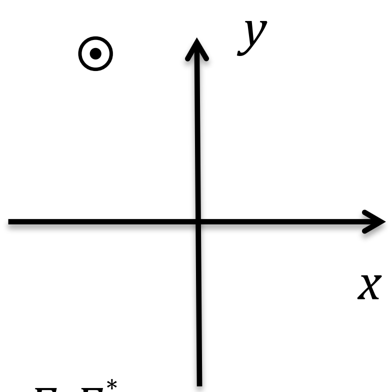


- Jones (complex) matrix:

$$\begin{pmatrix} A'_x e^{i\delta'_x} \\ A'_y e^{i\delta'_y} \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} A_x e^{i\delta_x} \\ A_y e^{i\delta_y} \end{pmatrix}$$

- Jones formalism uses electric fields
- Can account for interferences. Coherent superpositions.
- We do not measure the electric field of the waves in the visible

- Monochromatic plane wave: Stokes vector



$$E_x = A_x e^{i\delta_x} e^{i\tau}$$
$$E_y = A_y e^{i\delta_y} e^{i\tau}$$

$$\delta = \delta_x - \delta_y$$

$$I = A_x^2 + A_y^2$$
$$Q = A_x^2 - A_y^2$$
$$U = 2A_x A_y \cos\delta$$
$$V = 2A_x A_y \sin\delta$$

$$A_x^2 = E_x E_x^*$$
$$A_y^2 = E_y E_y^*$$
$$A_x A_y \cos\delta = \mathrm{Re}\left(E_x^* E_y\right)$$
$$A_x A_y \sin\delta = -\mathrm{Im}\left(E_x^* E_y\right)$$

- Quadratic on the fields → Intensities

$$I^2 = Q^2 + U^2 + V^2$$

- Linearly polarized light $\delta = m\pi \rightarrow \cos\delta = (-1)^m \quad \sin\delta = 0$

$$m = 0, \pm 1, \pm 2, \ldots$$

$$A_y = 0 \qquad \propto \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

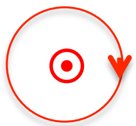
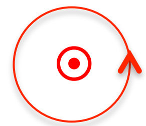
$$A_x = 0 \qquad \propto \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$$A_x = A_y \qquad \propto \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$A_x = -A_y \qquad \propto \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

- Circularly polarized light $\delta = m\frac{\pi}{2} \rightarrow \cos\delta = 0 \quad \sin\delta = \pm 1 \qquad A_x = A_y$

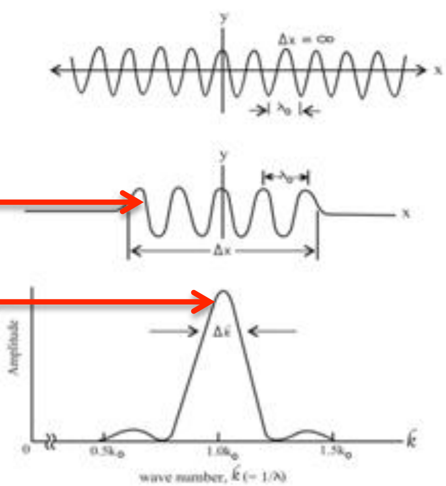
$$m = \pm 1, \pm 3, \ldots$$

$\propto \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$  $m = +1$ $\qquad \propto \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$  $m = -1$

- <u>Quasi-monochromatic plane waves</u>
- Coherence time τ_0 wave packets
- More than one frequency

$$\Delta\nu \sim \frac{1}{\tau_0} \ll \nu_0$$

- Electric dipole transitions $\Delta\nu \sim 10^{8} s^{-1} \quad \nu_0 \sim 10^{14} s^{-1}$
- Polarization ellipse changes after τ_0



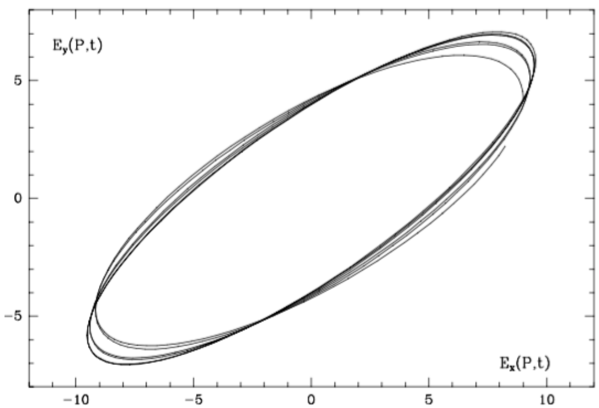
(a) Light Wave

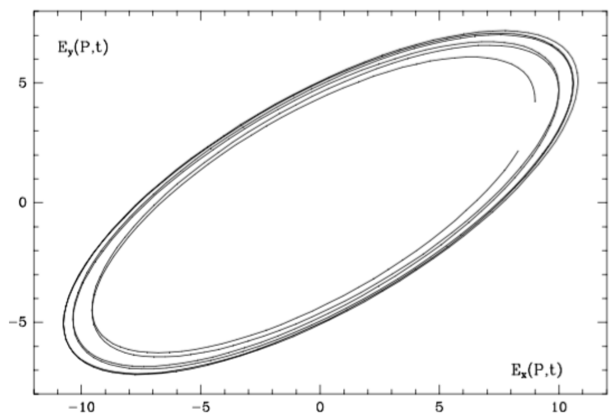
- Quasi-monochromatic plane wave

$$E_x(t) = A_x(t)e^{i\delta_x(t)}e^{i\tau}$$

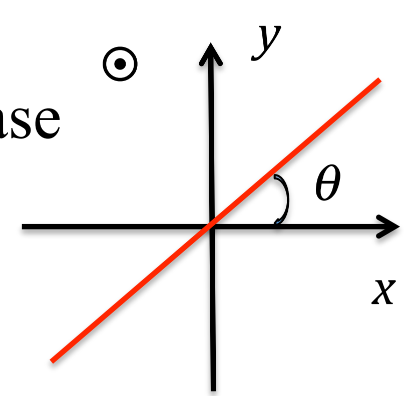
$$E_y(t) = A_y(t)e^{i\delta_y(t)}e^{i\tau}$$

- Polarization ellipse properties are a slow function of time





- <u>Operational definition of the Stokes parameters</u>
- Light goes through a retarder that introduces a phase difference ε to E_y
- Light goes through a linear polarizer at an angle θ



- Output light is linearly polarized with an E field:

$$E(t;\theta,\varepsilon) = E_x(t)\cos\theta + E_y(t)e^{i\varepsilon}\sin\theta$$

- We measure intensities averaging over $T \gg \tau_o$

$$\langle \ldots \rangle = \frac{1}{T}\int_0^T \ldots dt$$

$$I_{trans}(\theta,\varepsilon) = \left\langle E(t;\theta,\varepsilon)E^*(t;\theta,\varepsilon) \right\rangle = \frac{1}{T}\int_0^T E(t;\theta,\varepsilon)E^*(t;\theta,\varepsilon)dt$$

- Perform 6 measurements and define the Stokes parameters:

$$S_1 = I_{trans}(0,0)$$
$$S_2 = I_{trans}(\tfrac{\pi}{4},0)$$
$$S_3 = I_{trans}(\tfrac{\pi}{2},0)$$
$$S_4 = I_{trans}(\tfrac{3\pi}{4},0)$$
$$S_5 = I_{trans}(\tfrac{\pi}{4},\tfrac{\pi}{2})$$
$$S_6 = I_{trans}(\tfrac{3\pi}{4},\tfrac{\pi}{2})$$

$\longrightarrow$

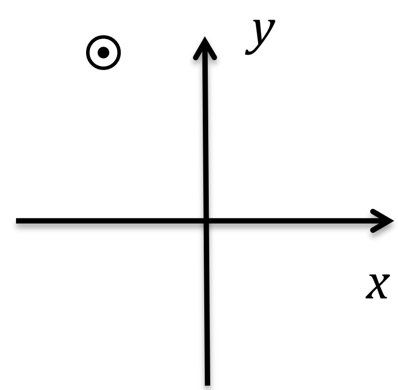
$$I := S_1 + S_3 (= S_2 + S_4 = S_5 + S_6)$$
$$Q := S_1 - S_3$$
$$U := S_2 - S_4$$
$$V := S_5 - S_6$$

- For a quasi-monochromatic plan wave

$$EE^{*} = A_x^2(t)\cos^2\theta + A_y^2(t)\sin^2\theta + 2A_x(t)A_y(t)\cos\theta\sin\theta\cos(\delta(t)-\varepsilon)$$

- One obtains:

$$I = \left\langle A_x^2 \right\rangle + \left\langle A_y^2 \right\rangle$$
$$Q = \left\langle A_x^2 \right\rangle - \left\langle A_y^2 \right\rangle$$
$$U = 2\left\langle A_x A_y \cos\delta \right\rangle$$
$$V = 2\left\langle A_x A_y \sin\delta \right\rangle$$



- Valid for all types of fields (not just monochromatic)
- Polarization degree ($\langle a \rangle^2 \le \langle a^2 \rangle$):

$$P_{pol} = \frac{\sqrt{Q^2 + U^2 + V^2}}{I} \le 1$$

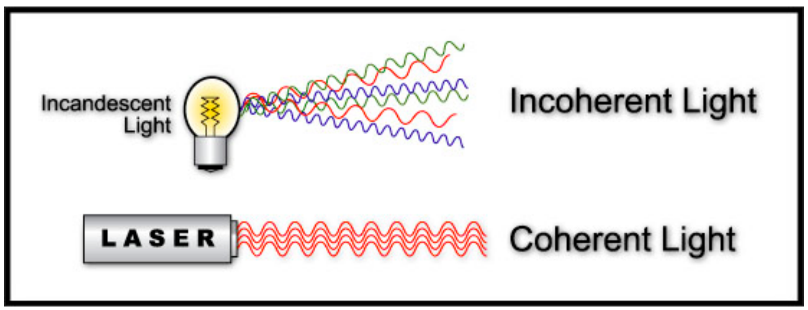
- Natural light (only Stokes):

$$\langle A_x^2 \rangle = \langle A_y^2 \rangle \qquad \left\langle A_x A_y \begin{matrix} \cos\delta \\ \sin\delta \end{matrix} \right\rangle = 0 \qquad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

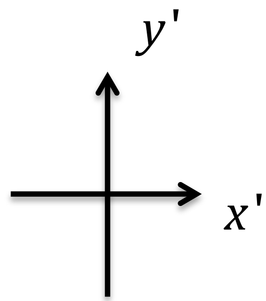
- Any Stokes vector can be decomposed in natural light and totally polarized light:

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \left(1 - P_{pol}\right) \begin{pmatrix} I \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} P_{pol} I \\ Q \\ U \\ V \end{pmatrix}$$

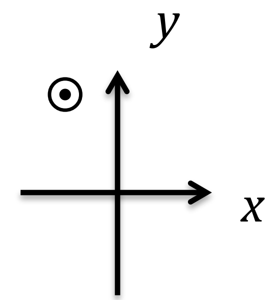
- Combining two incoherent light beams simply adds their Stokes vectors
- If the beams are coherent they add their Jones vectors.



- Mueller matrices: linear optical systems



$$\begin{pmatrix} I' \\ Q' \\ U' \\ V' \end{pmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix} \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix}$$

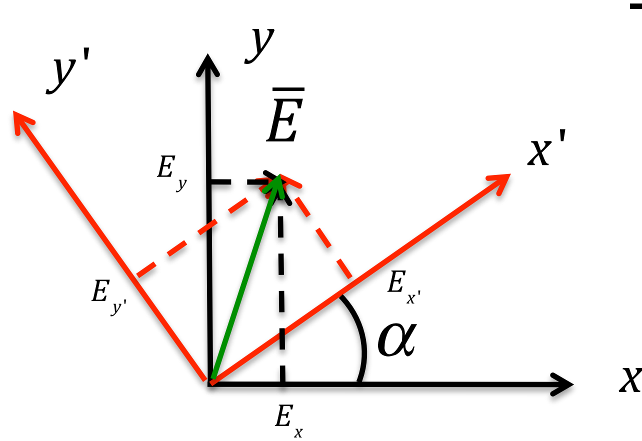


- <u>Note that there is an entrance and exit reference systems</u>

$$\bar{I}' = \bar{\bar{M}}\bar{I}$$

$$\bar{I}' = \bar{\bar{M}}_n \bar{\bar{M}}_{n-1} \ldots \bar{\bar{M}}_2 \bar{\bar{M}}_1 \bar{I}$$

- Some common Mueller matrices: <u>rotation</u>



$$E_{x'} = \left(E_x\right)_{x'} + \left(E_y\right)_{x'}$$

$$E_{y'} = \left(E_x\right)_{y'} + \left(E_y\right)_{y'}$$

- Which gives :

$$E_{x'} = E_x \cos\alpha + E_y \sin\alpha$$
$$E_{y'} = -E_x \sin\alpha + E_y \cos\alpha$$

$$E_x = A_x e^{i\delta_x} \qquad E_{x'} = A_{x'} e^{i\delta_{x'}}$$
$$E_y = A_y e^{i\delta_y} \qquad E_{y'} = A_{y'} e^{i\delta_{y'}}$$

- Output Stokes parameters:

$$A_{x'}^2 = E_{x'}E_{x'}^* = A_x^2 \cos^2\alpha + A_y^2 \sin^2\alpha + 2A_x A_y \cos\delta \cos\alpha \sin\alpha$$
$$A_{y'}^2 = E_{y'}E_{y'}^* = A_x^2 \sin^2\alpha + A_y^2 \cos^2\alpha - 2A_x A_y \cos\delta \cos\alpha \sin\alpha$$
$$A_{x'}A_{y'} \cos\delta' = \mathrm{Re}\left(E_{x'}^* E_{y'}\right) = (A_y^2 - A_x^2)\sin\alpha\cos\alpha + A_x A_y \cos\delta(\cos^2\alpha - \sin^2\alpha)$$
$$A_{x'}A_{y'} \sin\delta' = -\mathrm{Im}\left(E_{x'}^* E_{y'}\right) = A_x A_y \sin\delta$$

$$I' = \left\langle A_{x'}^2 \right\rangle + \left\langle A_{y'}^2 \right\rangle$$
$$Q' = \left\langle A_{x'}^2 \right\rangle - \left\langle A_{y'}^2 \right\rangle$$
$$U' = 2\left\langle A_{x'}A_{y'} \cos\delta' \right\rangle$$
$$V' = 2\left\langle A_{x'}A_{y'} \sin\delta' \right\rangle$$

$$Q' = Q\cos 2\alpha + U \sin 2\alpha$$
$$U' = -Q\sin 2\alpha + U \cos 2\alpha$$

$$c_{2\alpha} = \cos 2\alpha$$
$$s_{2\alpha} = \sin 2\alpha$$

- Mueller matrix of a rotation:

$$\begin{pmatrix} I' \\ Q' \\ U' \\ V' \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\alpha} & s_{2\alpha} & 0 \\ 0 & -s_{2\alpha} & c_{2\alpha} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \bar{\bar{R}}(\alpha)\bar{I} \qquad \bar{\bar{R}}(-\alpha) = \bar{\bar{R}}^{-1}(\alpha)$$

- <u>Mueller matrix of a generic process</u>:

$$\frac{E_{\perp}^{after}}{E_{||}^{after}} = Pe^{i\Delta} \frac{E_{\perp}^{before}}{E_{||}^{before}}$$

$$E_{\perp}^{after} = P_{\perp} e^{i\Delta_{\perp}} E_{\perp}^{before}$$

$$E_{||}^{after} = P_{||} e^{i\Delta_{||}} E_{||}^{before}$$

$$P = \frac{P_{\perp}}{P_{||}}$$

$$E_{||}^{before} = E_x = A_x e^{i\delta_x}$$

$$E_{\perp}^{before} = E_y = A_y e^{i\delta_y}$$

$$E_{||}^{after} = E_{x'} = A_{x'} e^{i\delta_{x'}}$$

$$E_{\perp}^{after} = E_{y'} = A_{y'} e^{i\delta_{y'}}$$

$$\Delta = \Delta_{\perp} - \Delta_{||} \qquad \delta = \delta_x - \delta_y \qquad \delta' = \delta_{x'} - \delta_{y'}$$

- Mueller matrix of a rotation:

$$A_{x'}^2 = E_{x'}E_{x'}^* = P_{\parallel}^2 A_x^2$$
$$A_{y'}^2 = E_{y'}E_{y'}^* = P_{\perp}^2 A_y^2$$
$$A_{x'}A_{y'}\cos\delta' = \mathrm{Re}\left(E_{x'}^* E_{y'}\right) = P_{\parallel}P_{\perp}A_x A_y \cos(\Delta-\delta)$$
$$A_{x'}A_{y'}\sin\delta' = -\mathrm{Im}\left(E_{x'}^* E_{y'}\right) = -P_{\parallel}P_{\perp}A_x A_y \sin(\Delta-\delta)$$

$$\left\langle A_x^2 \right\rangle = \frac{I+Q}{2}$$
$$\left\langle A_y^2 \right\rangle = \frac{I-Q}{2}$$

- $P_{\perp}, P_{\parallel}, \Delta_{\perp}, \Delta_{\parallel}$ are interaction constants independent of time:

$$I' = \left\langle A_{x'}^2 \right\rangle + \left\langle A_{y'}^2 \right\rangle = P_{\parallel}^2\left(I\left(\frac{1+P^2}{2}\right) + Q\left(\frac{1-P^2}{2}\right)\right)$$
$$Q' = \left\langle A_{x'}^2 \right\rangle - \left\langle A_{y'}^2 \right\rangle = P_{\parallel}^2\left(I\left(\frac{1-P^2}{2}\right) + Q\left(\frac{1+P^2}{2}\right)\right)$$
$$U' = 2\left\langle A_{x'}A_{y'}\cos\delta' \right\rangle = P_{\parallel}^2 P(U\cos\Delta + V\sin\Delta)$$
$$V' = 2\left\langle A_{x'}A_{y'}\sin\delta' \right\rangle = P_{\parallel}^2 P(-U\sin\Delta + V\cos\Delta)$$

- Mueller matrix of a generic process (reference frames depend on the process):

$$\begin{pmatrix} I' \\ Q' \\ U' \\ V' \end{pmatrix} = P_{\parallel}^{2} \begin{bmatrix} \frac{1+P^2}{2} & \frac{1-P^2}{2} & 0 & 0 \\ \frac{1-P^2}{2} & \frac{1+P^2}{2} & 0 & 0 \\ 0 & 0 & P\cos\Delta & P\sin\Delta \\ 0 & 0 & -P\sin\Delta & P\cos\Delta \end{bmatrix} \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix}$$

- Ignoring global transmission factors (that can otherwise be important):

$$\overline{\overline{M}}(P,\Delta) = \begin{bmatrix} \frac{1+P^2}{2} & \frac{1-P^2}{2} & 0 & 0 \\ \frac{1-P^2}{2} & \frac{1+P^2}{2} & 0 & 0 \\ 0 & 0 & P\cos\Delta & P\sin\Delta \\ 0 & 0 & -P\sin\Delta & P\cos\Delta \end{bmatrix}$$

- Refraction on a dielectric
- $\parallel = x$ is contained in the incidence-reflection plane
- $\perp = y$ is perpendicular

$$P = \cos(\theta_i - \theta_t)$$

$$\Delta = 0$$

$$\bar{\bar{M}}(P,\Delta) = \begin{bmatrix} \frac{1+\cos^2(\theta_i-\theta_t)}{2} & \frac{1-\cos^2(\theta_i-\theta_t)}{2} & 0 & 0 \\ \frac{1-\cos^2(\theta_i-\theta_t)}{2} & \frac{1+\cos^2(\theta_i-\theta_t)}{2} & 0 & 0 \\ 0 & 0 & \cos(\theta_i-\theta_t) & 0 \\ 0 & 0 & 0 & \cos(\theta_i-\theta_t) \end{bmatrix}$$

- Small refraction angles $\theta_t = \theta_i + \Delta\theta$:

$$\bar{\bar{M}}(P,\Delta) = \begin{bmatrix} 1 & \frac{\Delta\theta^2}{2} & 0 & 0 \\ \frac{\Delta\theta^2}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Normal incidence on a crystal $\theta_i = 0$ with optics axis at $\theta = \dfrac{\pi}{2}$
- $\| = x$ is parallel to optic axis and so is $\|' = x'$
- $\perp = y$ is perpendicular to optic axis

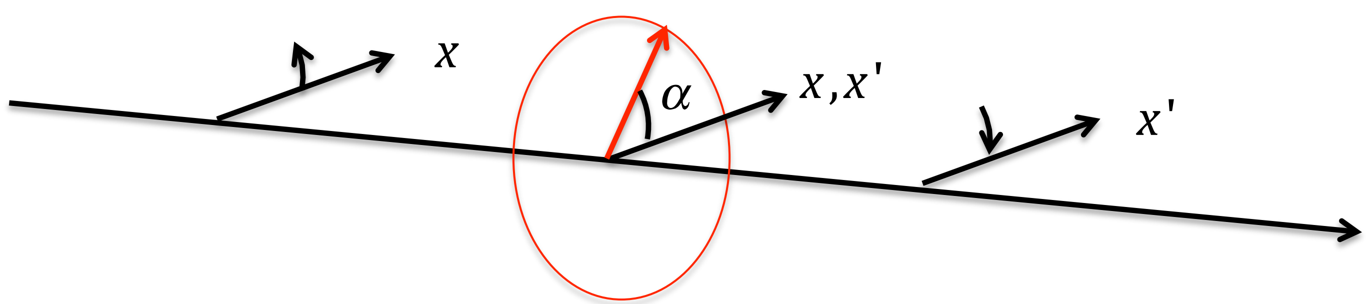
$$\Delta = \frac{2\pi d}{\lambda}(n_o - n_e) = -\frac{2\pi d}{\lambda}\delta$$

$$P = 1$$

$$\bar{\bar{M}}(1,\Delta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\Delta & \sin\Delta \\ 0 & 0 & -\sin\Delta & \cos\Delta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\frac{2\pi d}{\lambda}(n_o - n_e)) & \sin(\frac{2\pi d}{\lambda}(n_o - n_e)) \\ 0 & 0 & -\sin(\frac{2\pi d}{\lambda}(n_o - n_e)) & \cos(\frac{2\pi d}{\lambda}(n_o - n_e)) \end{bmatrix}$$

- Wave plate at an angle α :

$$\bar{\bar{M}}_\alpha(1,\Delta) = R(-\alpha)\bar{\bar{M}}(1,\Delta)R(\alpha)$$



- Wave plate at an angle α :

$$\bar{\bar{M}}_\alpha(1,\Delta)=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\alpha}^2+s_{2\alpha}^2\cos\Delta & c_{2\alpha}s_{2\alpha}(1-\cos\Delta) & -s_{2\alpha}\sin\Delta \\ 0 & c_{2\alpha}s_{2\alpha}(1-\cos\Delta) & s_{2\alpha}^2+c_{2\alpha}^2\cos\Delta & c_{2\alpha}\sin\Delta \\ 0 & s_{2\alpha}\sin\Delta & -c_{2\alpha}\sin\Delta & \cos\Delta \end{bmatrix} \qquad \begin{array}{l} c_{2\alpha}=\cos 2\alpha \\ s_{2\alpha}=\sin 2\alpha \end{array}$$

- Dichroism (Polaroids):

$$P=e^{-\beta(\kappa_\perp-\kappa_\parallel)} \qquad \beta=\frac{\varpi}{c}nd$$

$$\Delta=0 \qquad \kappa_\perp \gg \kappa_\parallel \rightarrow P \ll 1$$

$$\bar{\bar{M}}(P,0)=\begin{bmatrix} \frac{1+P^2}{2} & \frac{1-P^2}{2} & 0 & 0 \\ \frac{1-P^2}{2} & \frac{1+P^2}{2} & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix} \approx \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Partial polarizer axis at at an angle α :

$$\bar{\bar{M}}_\alpha(P,0)=\begin{bmatrix} \frac{1+P^2}{2} & c_{2\alpha}\frac{1-P^2}{2} & s_{2\alpha}\frac{1-P^2}{2} & 0 \\ c_{2\alpha}\frac{1-P^2}{2} & c_{2\alpha}^2\frac{1+P^2}{2}+s_{2\alpha}^2 P & c_{2\alpha}s_{2\alpha}\frac{1+P^2}{2}-s_{2\alpha}c_{2\alpha}P & 0 \\ s_{2\alpha}\frac{1-P^2}{2} & c_{2\alpha}s_{2\alpha}\frac{1+P^2}{2}-s_{2\alpha}c_{2\alpha}P & s_{2\alpha}^2\frac{1+P^2}{2}+c_{2\alpha}^2 P & 0 \\ 0 & 0 & 0 & P \end{bmatrix}$$

- Ideal polarizer at an angle α :

$$\bar{\bar{M}}_\alpha(0,0)=\frac{1}{2}\begin{bmatrix} 1 & c_{2\alpha} & s_{2\alpha} & 0 \\ c_{2\alpha} & c_{2\alpha}^2 & c_{2\alpha}s_{2\alpha} & 0 \\ s_{2\alpha} & c_{2\alpha}s_{2\alpha} & s_{2\alpha}^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

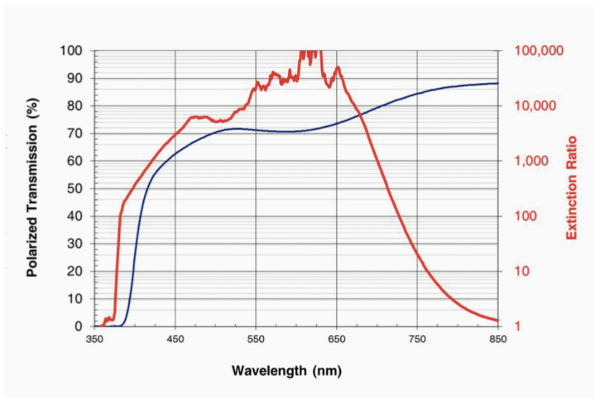
- 2 identical partial polarizer axis at at zero degrees :

$$\bar{\bar{M}}_0(P,0)\bar{\bar{M}}_0(P,0)=\begin{bmatrix} \frac{1+P^4}{2} & \frac{1-P^4}{2} & 0 & 0 \\ \frac{1-P^4}{2} & \frac{1+P^4}{2} & 0 & 0 \\ 0 & 0 & P^2 & 0 \\ 0 & 0 & 0 & P^2 \end{bmatrix} \rightarrow I'_{\parallel} \propto \left(\frac{1+P^4}{2}\right) I$$

- 2 identical partial polarizer axis at at 90 degrees :

$$\bar{\bar{M}}_{\frac{\pi}{2}}(P,0)\bar{\bar{M}}_0(P,0)=\begin{bmatrix} P^2 & 0 & 0 & 0 \\ 0 & P^2 & 0 & 0 \\ 0 & 0 & P^2 & 0 \\ 0 & 0 & 0 & P^2 \end{bmatrix} \rightarrow I'_{\perp} \propto P^2 I$$

$$\frac{I'_{\parallel}}{I'_{\perp}} = \frac{P^2}{\left(\frac{1+P^4}{2}\right)} \approx 2P^2 \rightarrow$$



- Mueller matrix of a mirror reflection:
- $\| = x$ parallel to incidence reflection plane
- $\perp = y$ perpendicular to incidence reflection plane

$$P^2 = \frac{1}{\chi^2} = \frac{f^2 + g^2 + 2f\sin\theta_i \tan\theta_i + \sin^2\theta_i \tan^2\theta_i}{f^2 + g^2 - 2f\sin\theta_i \tan\theta_i + \sin^2\theta_i \tan^2\theta_i} \qquad \tan\Delta = \frac{2g\sin\theta_i \tan\theta_i}{\sin^2\theta_i \tan^2\theta_i - (f^2 + g^2)}$$

$$\bar{\bar{M}}(P,\Delta) = \begin{bmatrix} \frac{1+P^2}{2} & \frac{1-P^2}{2} & 0 & 0 \\ \frac{1-P^2}{2} & \frac{1+P^2}{2} & 0 & 0 \\ 0 & 0 & P\cos\Delta & P\sin\Delta \\ 0 & 0 & -P\sin\Delta & P\cos\Delta \end{bmatrix}$$

- DKIST primary and secondary <u>off-axis</u> configuration
- $\| = x$ incidence-reflection plane of principal ray at M1
- Same as incidence-reflection plane of principal ray at M2
- Aluminum coating, treat M1 and M2 as flat mirrors

- $\theta_i^1 = 28.1^o$ and $\theta_i^2 = 23.7^o$ at 4000 Å:

$$\bar{\bar{M}}_{DKIST}(P,\Delta) = \begin{bmatrix} 1.00 & -0.018 & 0 & 0 \\ -0.018 & 1.00 & 0 & 0 \\ 0 & 0 & 0.984 & -0.178 \\ 0 & 0 & 0.178 & 0.984 \end{bmatrix}$$

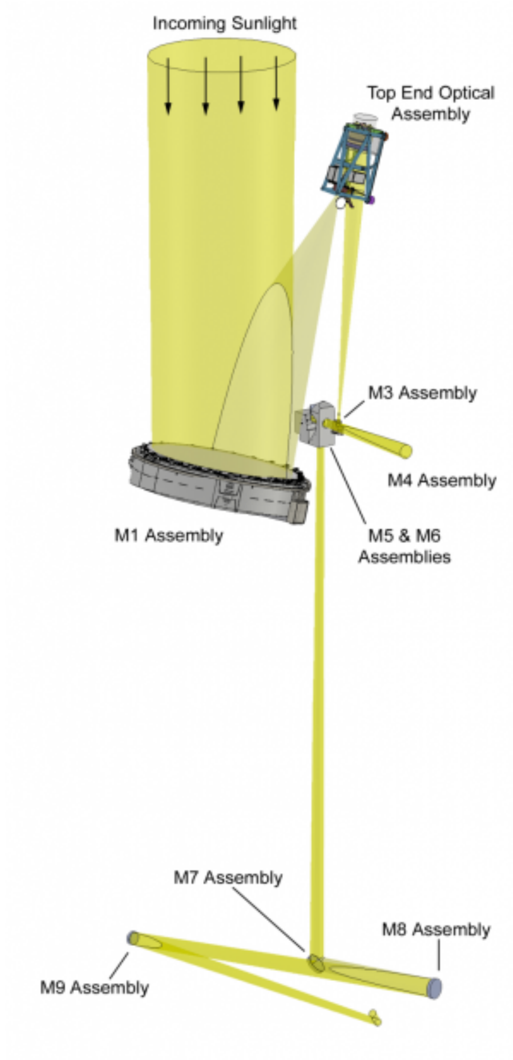
- At 6300 Å:

$$\bar{\bar{M}}_{DKIST}(P,\Delta) = \begin{bmatrix} 1.00 & -0.020 & 0 & 0 \\ -0.020 & 1.00 & 0 & 0 \\ 0 & 0 & 0.993 & -0.114 \\ 0 & 0 & 0.114 & 0.993 \end{bmatrix}$$

- At 15000 Å (1.5 μm):

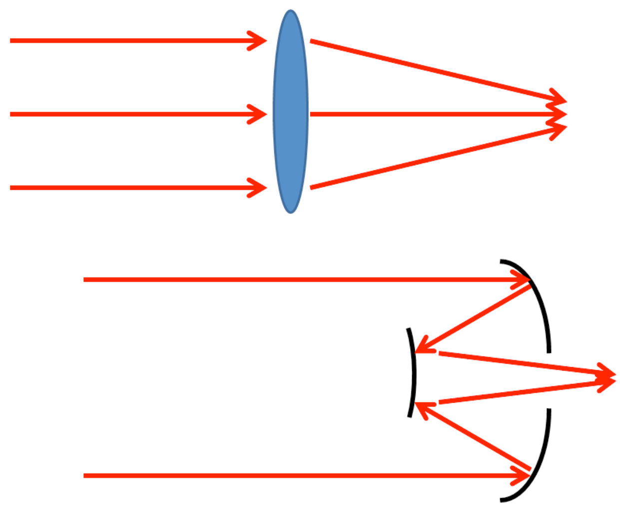
$$\bar{\bar{M}}_{DKIST}(P,\Delta) = \begin{bmatrix} 1.00 & -0.0057 & 0 & 0 \\ -0.0057 & 1.00 & 0 & 0 \\ 0 & 0 & 0.998 & -0.0575 \\ 0 & 0 & 0.0575 & 0.998 \end{bmatrix}$$

http://refractiveindex.info/



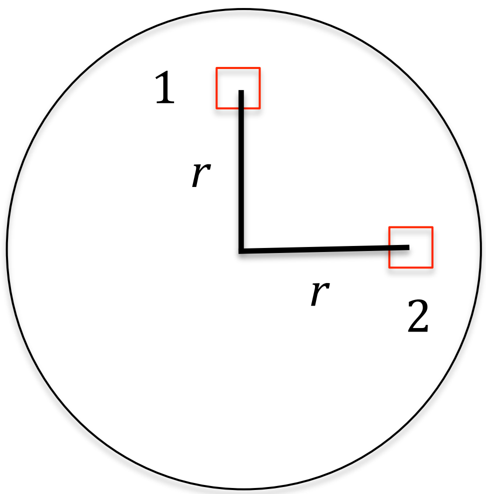
Much better in the IR

- <u>Mueller matrix of on axis image forming mirrors (and lenses)</u>
- At the focal plane we bring together coherent rays
- Rays interfere and form an image
- Stokes formalism cannot describe the interference process
- Need to use electric vectors (Jones)

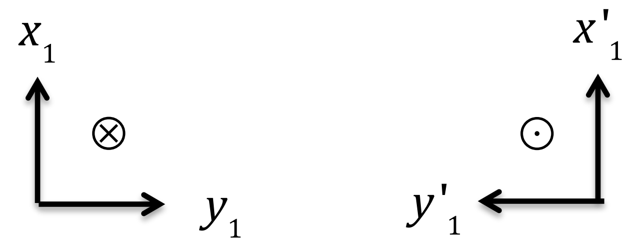


- Symmetry along the optical axis
- It is not even evident where to put the $\parallel = x$ axis
- Polarization is all about breaking symmetries

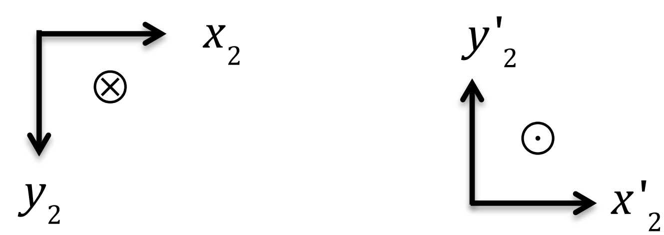
- <u>Mueller matrix of on-axis image forming mirrors</u>
- Select two small mirror sections at same radial distance and at 90 degrees of each other (1 & 2)



- Select ray hitting area 1 and reflected to focal plane
- Reflection-plane is $\| = x$

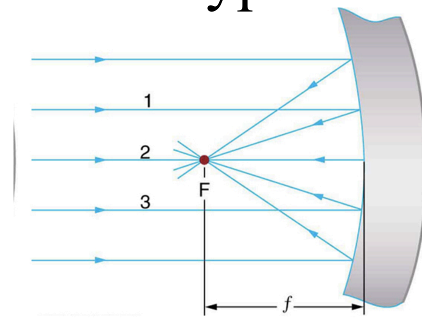


- Select ray hitting area 2 and reflected to focal plane
- Reflection-plane is $\| = x$



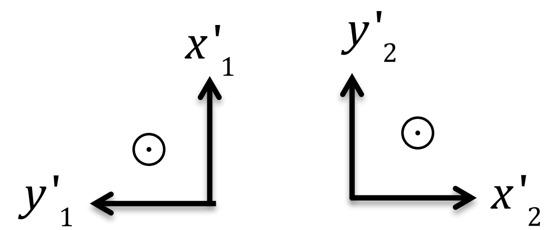
- We search for a relation at the focal plane of the type:

$$\frac{E_{y'}}{E_{x'}} = P_f e^{i\Delta_f} \frac{E_y}{E_x}$$



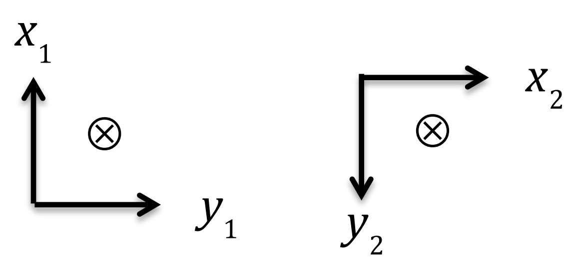
- Select reference frames of ray 1 for the global Mueller matrix
- Make them interfere at focus

$$E_{x'} = E_{x'_1} + E_{y'_2}$$
$$E_{y'} = E_{y'_1} - E_{x'_2}$$



- The input wavefront is homogenous making:

$$E_{x_1} := E_x = -E_{y_2} \quad E_{y_1} := E_y = E_{x_2}$$



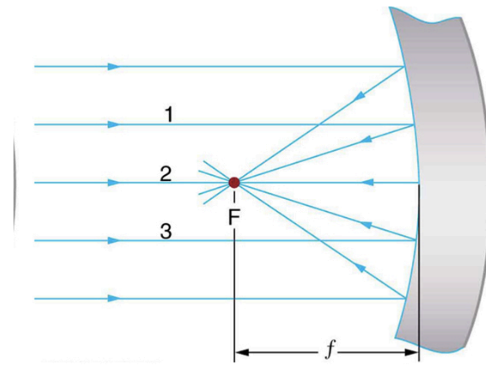
- Mirror has constant properties over the surface
- Rays 1 and 2 are at same distance from center $\theta(r)=\theta_i^1=\theta_i^2$

$$\frac{E_{y'_1}}{E_{x'_1}}=P_1e^{i\Delta_1}\frac{E_{y_1}}{E_{x_1}}=Pe^{i\Delta}\frac{E_y}{E_x}$$

$$P=P_1=P_2$$
$$\Delta=\Delta_1=\Delta_2$$

$$\frac{E_{y'_2}}{E_{x'_2}}=P_2e^{i\Delta_2}\frac{E_{y_2}}{E_{x_2}}=-Pe^{i\Delta}\frac{E_x}{E_y}$$

$$E_{x_1}:=E_x=-E_{y_2}\qquad E_{y_1}:=E_y=E_{x_2}$$



- Because 1 and 2 are equivalent, we also have these relations between what happens in x and y :

$$\frac{E_{x'_1}}{E_{x_1}}=\frac{E_{x'_2}}{E_{x_2}}\rightarrow\frac{E_{x'_2}}{E_{x'_1}}=\frac{E_y}{E_x}\qquad\qquad\frac{E_{y'_1}}{E_{y_1}}=\frac{E_{y'_2}}{E_{y_2}}\rightarrow\frac{E_{y'_2}}{E_{y'_1}}=-\frac{E_x}{E_y}$$

- We search for a relation at the focal plane of the type:

$$E_{x'} = E_{x'_1} + E_{y'_2}$$

$$E_{y'} = E_{y'_1} - E_{x'_2}$$

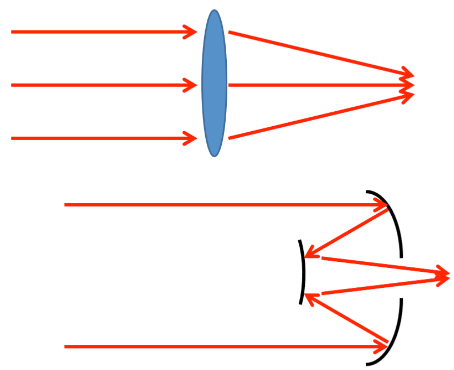
$$E_{x'} = E_{x'_1}\left(1 + \frac{E_{y'_2}}{E_{x'_1}}\right) = E_{x'_1}\left(1 + \frac{E_{y'_2}}{E_{x'_2}}\frac{E_{x'_2}}{E_{x'_1}}\right) = E_{x'_1}\left(1 - Pe^{i\Delta}\right)$$

$$E_{y'} = E_{x'_1}\left(\frac{E_{y'_1}}{E_{x'_1}} - \frac{E_{x'_2}}{E_{x'_1}}\right) = E_{x'_1}\left(Pe^{i\Delta}\frac{E_y}{E_x} - \frac{E_{x_2}}{E_{x_1}}\right) = E_{x'_1}\frac{E_y}{E_x}\left(Pe^{i\Delta} - 1\right)$$

- Taking the ratio:

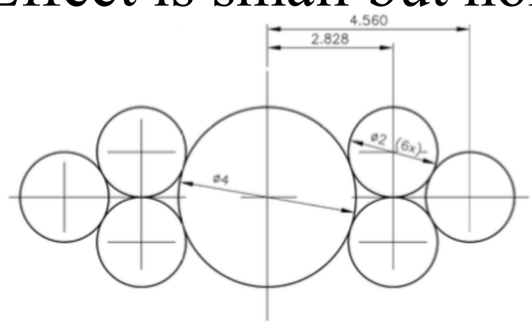
$$\frac{E_{y'}}{E_{x'}} = -\frac{E_y}{E_x} \rightarrow \quad P_f = 1 \quad \Delta = \pi$$

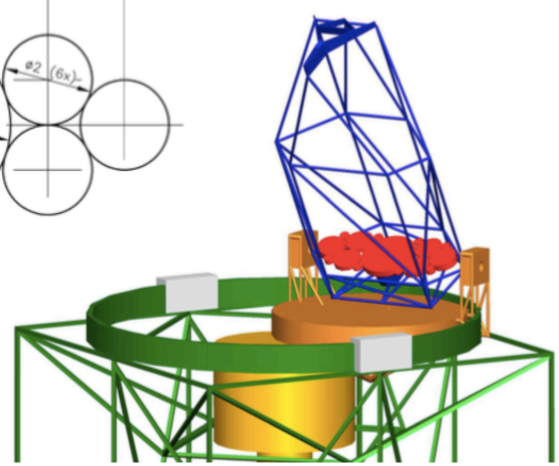
- Mueller matrix of on axis image forming mirrors

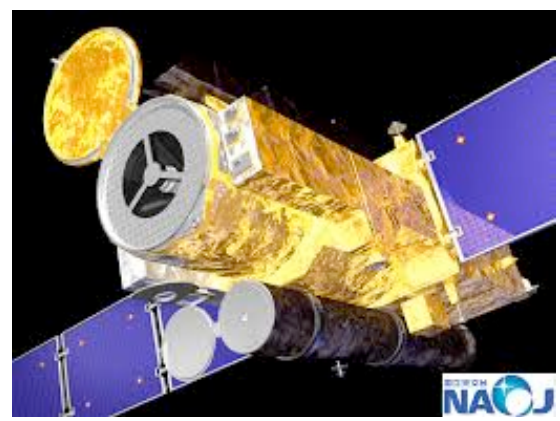


$$\bar{\bar{M}}_{mirror} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad \bar{\bar{M}}_{lens} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Compensation occurs from patches at 90 degrees
- Similarly one can prove a lens has as Muller matrix the identity
- This is only true on-axis, not off-axis (other points of the FOV)
- Effect is small but non-negligible

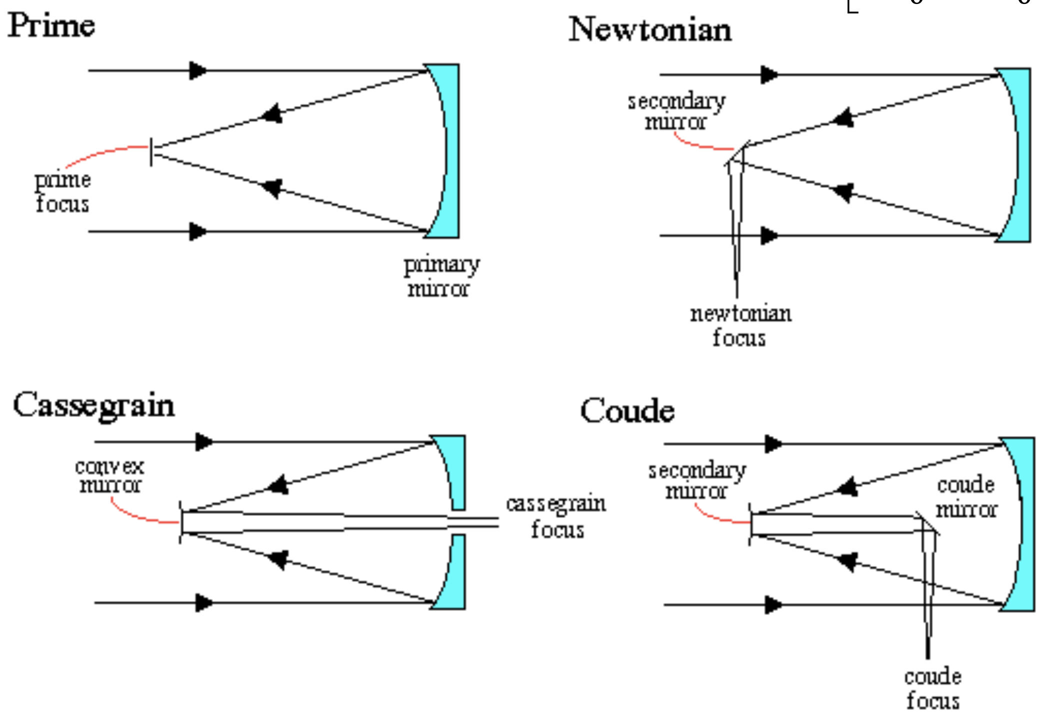






$$\bar{\bar{M}}_{Prime} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\bar{\bar{M}}_{Newton}(P,\Delta) = \begin{bmatrix} \frac{1+P^2}{2} & \frac{1-P^2}{2} & 0 & 0 \\ \frac{1-P^2}{2} & \frac{1+P^2}{2} & 0 & 0 \\ 0 & 0 & -P\cos\Delta & P\sin\Delta \\ 0 & 0 & -P\sin\Delta & -P\cos\Delta \end{bmatrix}$$



$$\bar{\bar{M}}_{Cassegrain} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\bar{\bar{M}}_{Coude}(P,\Delta) = \begin{bmatrix} \frac{1+P^2}{2} & \frac{1-P^2}{2} & 0 & 0 \\ \frac{1-P^2}{2} & \frac{1+P^2}{2} & 0 & 0 \\ 0 & 0 & P\cos\Delta & P\sin\Delta \\ 0 & 0 & -P\sin\Delta & P\cos\Delta \end{bmatrix}$$

III. Modern spectropolartimeters

http://adsabs.harvard.edu/abs/1999ASPC..184....3C

High Resolution Spectropolarimetry and Magnetography
Collados, M

http://adsabs.harvard.edu/abs/2000ApOpt..39.1637D

Optimum Modulation and Demodulation Matrices for Solar Polarimetry
del Toro Iniesta, Jose Carlos; Collados, Manuel

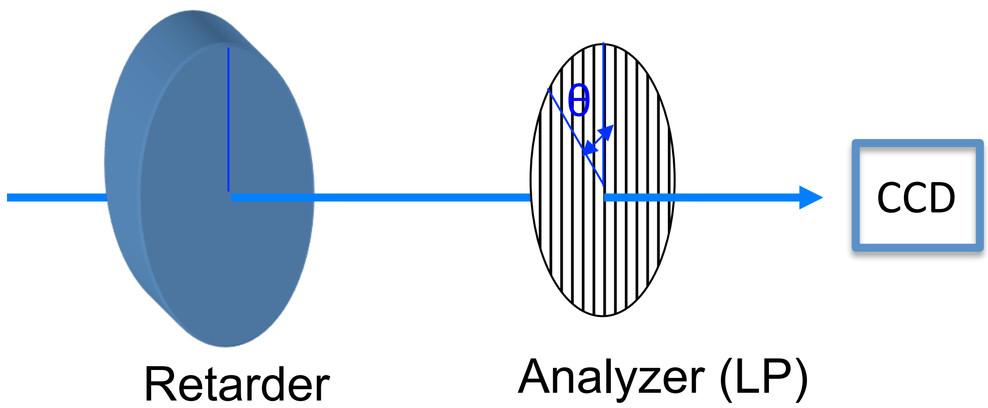
http://adsabs.harvard.edu/abs/2008ApOpt..47.2541R

Error propagation in polarimetric demodulation
Ramos, A. Asensio; Collados, M.

http://adsabs.harvard.edu/abs/2010ApOpt..49.3580T

Wavelength-diverse polarization modulators for Stokes polarimetry
Tomczyk, Steven; Casini, Roberto; de Wijn, Alfred G.; Nelson, Peter G.

- <u>Operational definition of the Stokes parameters</u>:



$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix}$$

$$\varepsilon = \frac{\pi}{2} \qquad \theta = \left(0, \ \frac{\pi}{4}, \ \frac{\pi}{2}, \ \frac{3\pi}{4} \right)$$

$$S_1 = I_{trans}(0,0)$$
$$S_2 = I_{trans}(\tfrac{\pi}{4},0)$$
$$S_3 = I_{trans}(\tfrac{\pi}{2},0)$$
$$S_4 = I_{trans}(\tfrac{3\pi}{4},0)$$
$$S_5 = I_{trans}(\tfrac{\pi}{4},\tfrac{\pi}{2})$$
$$S_6 = I_{trans}(\tfrac{3\pi}{4},\tfrac{\pi}{2})$$

$$I = S_1 + S_3$$
$$Q = S_1 - S_3$$
$$U = S_2 - S_4$$
$$V = S_5 - S_6$$

- 6 measurements for 4 parameters
- Measure Q, but no U or V
- Retarder is out and brought in for last 2
- Analyzer rotates into 4 angles → changing polarization on the detector

<u>Can we do better?</u>

- Analyzer: $\theta = \left(0, \quad \frac{\pi}{4}, \quad \frac{\pi}{2}, \quad \frac{3\pi}{4} \right)$

$$\bar{\bar{M}}_0(0,0) = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \bar{\bar{M}}_{\frac{\pi}{4}}(0,0) = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \bar{\bar{M}}_{\frac{\pi}{2}}(0,0) = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \bar{\bar{M}}_{\frac{3\pi}{4}}(0,0) = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Analyzer + retarder: $\theta = \left(\frac{\pi}{4}, \quad \frac{3\pi}{4} \right) \qquad \varepsilon = \frac{\pi}{2}$

$$\bar{\bar{M}}_{\frac{\pi}{4}}(0,0)\bar{\bar{M}}_0(1,\frac{\pi}{2}) = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \bar{\bar{M}}_{\frac{3\pi}{4}}(0,0)\bar{\bar{M}}_0(1,\frac{\pi}{2}) = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Linear combinations between Stokes and intensity measures

$$\begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{pmatrix} = \frac{1}{2}\begin{bmatrix} +1 & +1 & 0 & 0 \\ +1 & 0 & +1 & 0 \\ +1 & -1 & 0 & 0 \\ +1 & 0 & -1 & 0 \\ +1 & 0 & 0 & +1 \\ +1 & 0 & 0 & -1 \end{bmatrix} \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \frac{1}{2}\begin{pmatrix} I+Q \\ I+U \\ I-Q \\ I-U \\ I+V \\ I-V \end{pmatrix}$$

- The matrix relating measurements and Stokes is called <u>modulation matrix</u>
- Made of first rows of Mueller matrices, but not a Mueller matrix itself

$$\mathbf{X} = \begin{bmatrix} +1 & +1 & 0 & 0 \\ +1 & 0 & +1 & 0 \\ +1 & -1 & 0 & 0 \\ +1 & 0 & -1 & 0 \\ +1 & 0 & 0 & +1 \\ +1 & 0 & 0 & -1 \end{bmatrix}$$

- $\mathbf{X}$ is $n\times4$, with n the number of modulation states
- $\mathbf{X}$ is always of the form (assuming transmission factors are the same, ideal behavior):

$$\mathbf{X} = \begin{bmatrix} 1 & . & . & . \\ 1 & . & . & . \\ 1 & . & . & . \\ 1 & . & . & . \\ . & & & \end{bmatrix} \qquad \mathbf{X} = \begin{bmatrix} m_{1,11} & . & . & . \\ m_{2,11} & . & . & . \\ m_{3,11} & . & . & . \\ m_{4,11} & . & . & . \\ . & & & \\ m_{n,11} & & & \end{bmatrix} \quad m_{i,11} \approx 1$$

- Modulation matrix $\mathbf{X}$, input Stokes vector $\overline{I}_{\odot}$ and vector of measured intensities (modulation states) $\overline{S}$ are related by:

$$\overline{S} = \mathbf{X}\overline{I}_{\odot}$$

- The demodulation matrix is defined as:

$$\overline{I}_{\odot} = \mathbf{D}\overline{S} \quad (n=4) \rightarrow \overline{I}_{\odot} = \mathbf{X}^{-1}\overline{S}$$

- **D** is 4×*n* with *n* the number of modulation states
- Not unique if $n \neq 4$.
- <u>**D** that maximizes signal-to-noise of the Stokes parameters and that fulfills **DX=1** (4×4 identity).</u>

$$\overline{I}_{\odot} = \begin{bmatrix} I & Q & U & V \end{bmatrix}^T = \begin{bmatrix} I_1 & I_2 & I_3 & I_4 \end{bmatrix}^T$$

$$I_i = \sum_{j=1}^{n} D_{ij} S_j \qquad \sigma_{I_i}^2 = \sum_{j=1}^{n} \left(\frac{\partial I_i}{\partial S_j} \right)^2 \sigma_{S_j}^2$$

- Error propagation gives (σ is noise in each intensity S_j measurement, all assumed equal):

$$\sigma_{I_i} = \sigma \left(\sum_{j=1}^{n} D_{ij}^2 \right)^{1/2} = \frac{\sigma}{\sqrt{n}} \left(n \sum_{j=1}^{n} D_{ij}^2 \right)^{1/2} = \frac{\sigma}{\varepsilon_i \sqrt{n}}$$

- Combining n measurements reduces de error by $\sqrt{n}$
- The other factor characterizes the polarimetric scheme

$$\boxed{\varepsilon_i = \left(n \sum_{j=1}^{n} D_{ij}^2 \right)^{-1/2}}$$

$$\varepsilon_I \leq 1; \;\; \varepsilon_Q^2 + \varepsilon_U^2 + \varepsilon_V^2 \leq 1$$

$$\varepsilon_Q^{\max} = \varepsilon_U^{\max} = \varepsilon_V^{\max} = \frac{1}{\sqrt{3}} = 0.577$$

- Polarimetric efficiencies. The larger ε_i the smaller σ_{I_i}
- Define **D** so that efficiencies are optimized (not maximized)
- **D** is the Moore-Penrose pseudoinverse

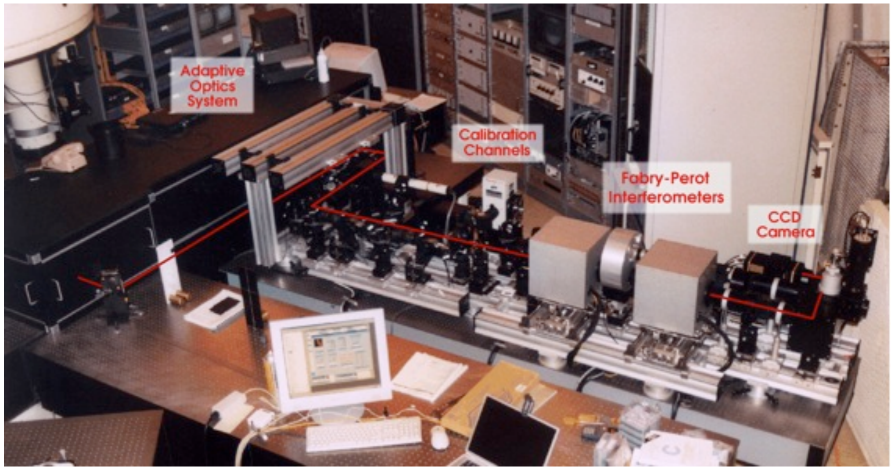
$$\mathbf{D} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T$$

- For our operational definition: http://comnuan.com/cmnn0100f/

$$\mathbf{X} = \begin{bmatrix} +1 & +1 & 0 & 0 \\ +1 & 0 & +1 & 0 \\ +1 & -1 & 0 & 0 \\ +1 & 0 & -1 & 0 \\ +1 & 0 & 0 & +1 \\ +1 & 0 & 0 & -1 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 0.1667 & 0.1667 & 0.1667 & 0.1667 & 0.1667 & 0.1667 \\ 0.5 & 0 & -0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & -0.5 \end{bmatrix}$$

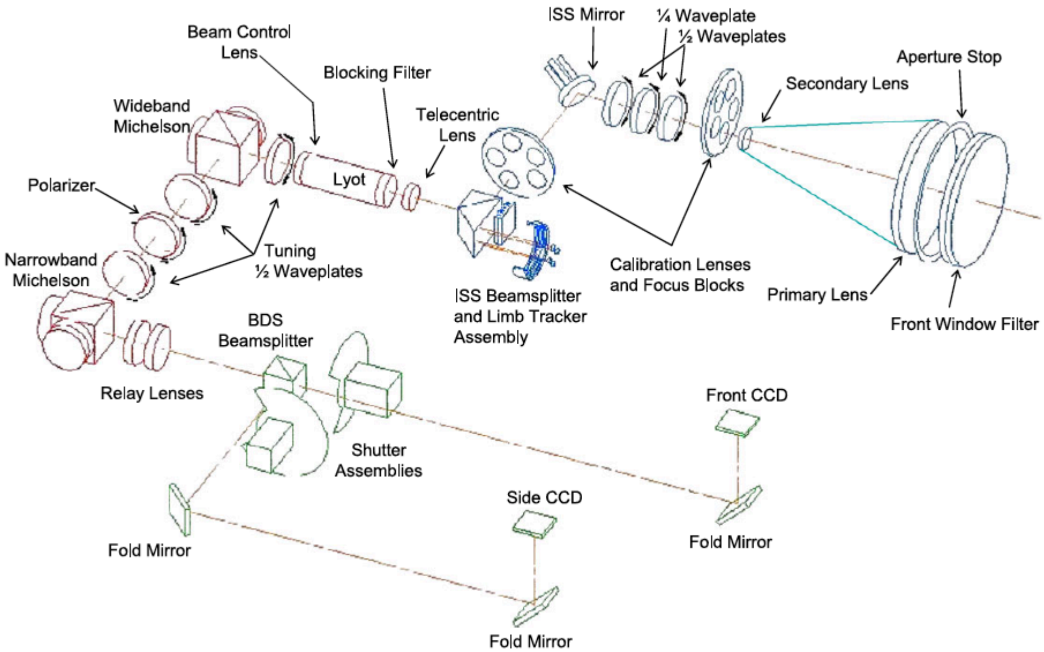
- Efficiencies: $\begin{pmatrix} \varepsilon_I & \varepsilon_Q & \varepsilon_U & \varepsilon_V \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$

- IBIS at DST
- No rotating elements
- Variable retarders instead
- Zeros in **D** mean frames do not contribute

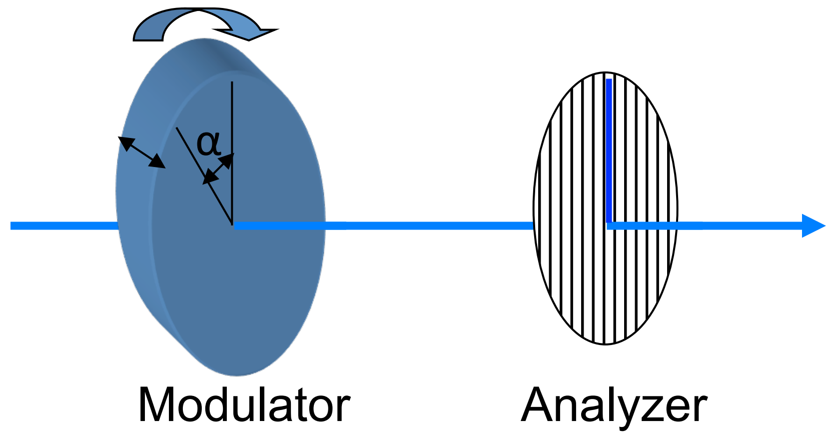


- SDO/HMI
- Rotating elements
- Three waveplates $\frac{\lambda}{2}; \frac{\lambda}{4}; \frac{\lambda}{2}$
- Generates "pure" states

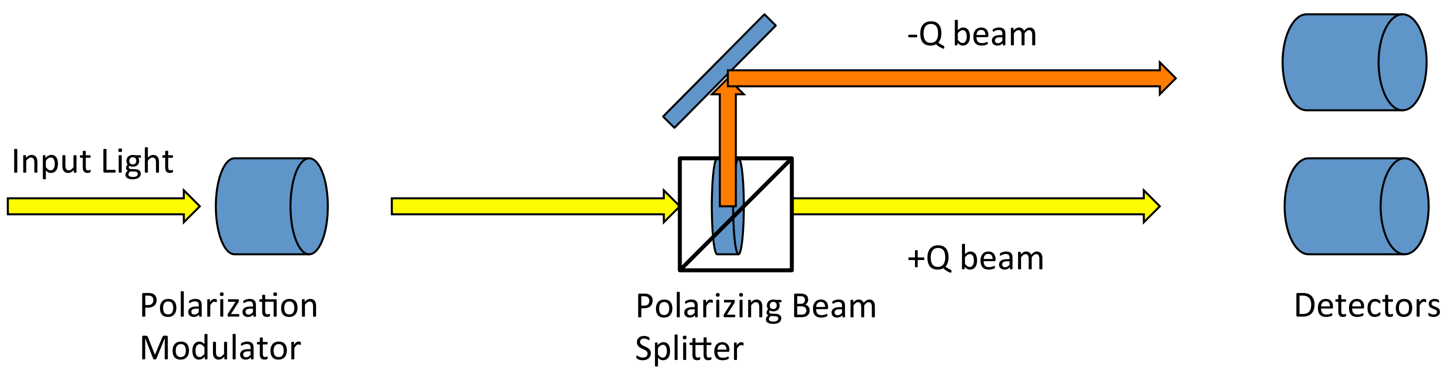
$I \pm Q \quad I \pm U \quad I \pm V$



- Rotating retarder plus analyzer:



- Rotating retarder plus analyzer (0, 90 degrees):



- Rotating retarder plus analyzer (0, 90 degrees):

$$\bar{\bar{M}}_{0,\frac{\pi}{2}}(0,0)\bar{\bar{M}}_{\alpha(t)}(1,\Delta)=\frac{1}{2}\begin{bmatrix} 1 & \pm1 & 0 & 0 \\ \pm1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\alpha(t)}^2+s_{2\alpha(t)}^2\cos\Delta & c_{2\alpha(t)}s_{2\alpha(t)}(1-\cos\Delta) & -s_{2\alpha(t)}\sin\Delta \\ 0 & c_{2\alpha(t)}s_{2\alpha(t)}(1-\cos\Delta) & s_{2\alpha(t)}^2+c_{2\alpha(t)}^2\cos\Delta & c_{2\alpha(t)}\sin\Delta \\ 0 & s_{2\alpha(t)}\sin\Delta & -c_{2\alpha(t)}\sin\Delta & \cos\Delta \end{bmatrix}$$

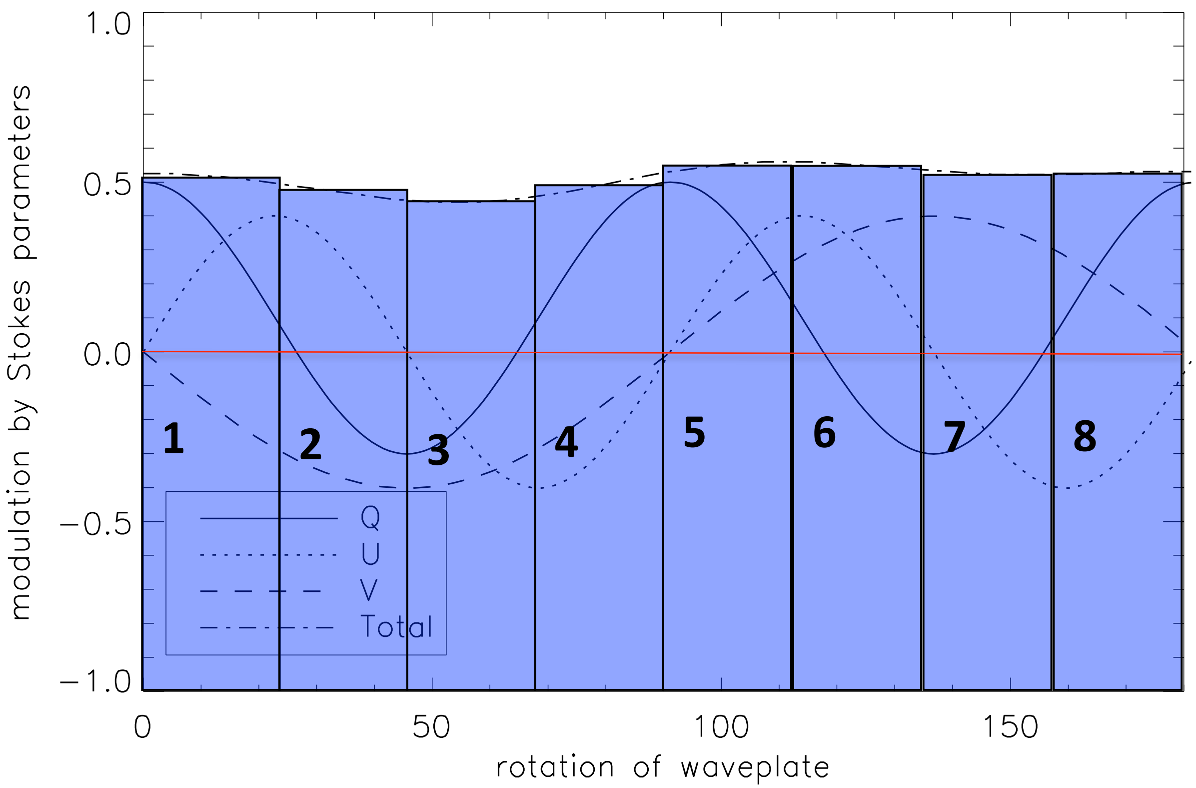
$$\bar{I}_D^+=\bar{\bar{M}}_0(0,0)\bar{\bar{M}}_{\alpha(t)}(1,\Delta)\bar{I}_\odot=\frac{1}{2}\begin{bmatrix} 1 & c_{2\alpha(t)}^2+s_{2\alpha(t)}^2\cos\Delta & c_{2\alpha(t)}s_{2\alpha(t)}(1-\cos\Delta) & -s_{2\alpha(t)}\sin\Delta \\ 1 & c_{2\alpha(t)}^2+s_{2\alpha(t)}^2\cos\Delta & c_{2\alpha(t)}s_{2\alpha(t)}(1-\cos\Delta) & -s_{2\alpha(t)}\sin\Delta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix}$$

$$\bar{I}_D^+=\frac{1}{2}(I_\odot+\left[c_{2\alpha(t)}^2+s_{2\alpha(t)}^2\cos\Delta\right]Q_\odot+\left[c_{2\alpha(t)}s_{2\alpha(t)}(1-\cos\Delta)\right]U_\odot-\left[s_{2\alpha(t)}\sin\Delta\right]V_\odot)\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

- Rotating retarder $\alpha(t)=\omega t$:

$$I_D^+=\frac{1}{2}(I_\odot+\left[(1+\cos\Delta)+\cos 4\omega t\cdot(1-\cos\Delta)\right]\frac{Q_\odot}{2}+\left[\sin 4\omega t\cdot(1-\cos\Delta)\right]\frac{U_\odot}{2}-\left[\sin 2\omega t\cdot\sin\Delta\right]V_\odot)$$

- Rotating retarder:



Parameter	Combination
I	1+2+3+4+5+6+7+8
Q	1-2-3+4+5-6-7+8
U	1+2-3-4+5+6-7-8
V	-1-2-3-4+5+6+7+8

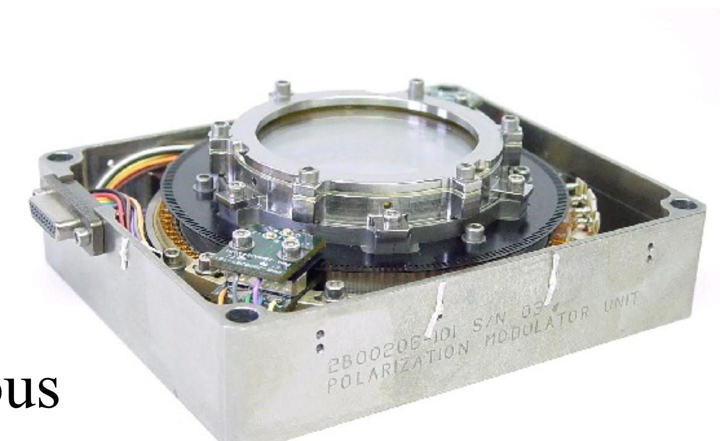
- Equal modulation amplitudes for $\Delta = 127^o$
- *Q* has a DC component
- *Q* and *U* modulated at twice the frequency of *V*
- *Q* and *U* phase shift
- Forces 8 *S* measurements (*n*=8)

- For $\Delta = 127^o$ integrating every 8th interval:

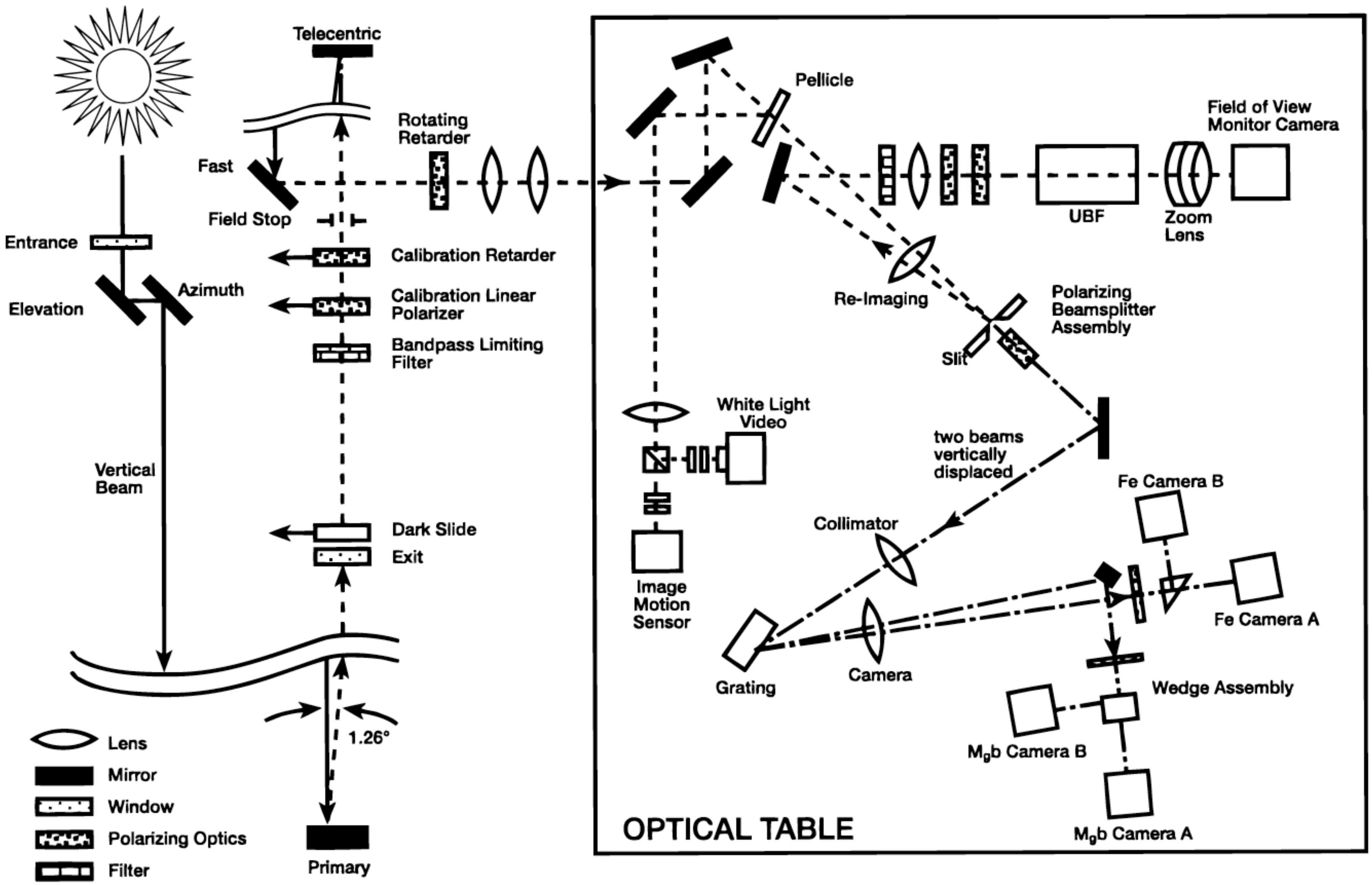
$$\mathbf{X} = \begin{bmatrix} +1 & +0.71 & +0.51 & -0.30 \\ +1 & -0.31 & +0.51 & -0.72 \\ +1 & -0.31 & -0.51 & -0.72 \\ +1 & +0.71 & -0.51 & -0.30 \\ +1 & +0.71 & +0.51 & +0.30 \\ +1 & -0.31 & +0.51 & +0.72 \\ +1 & -0.31 & -0.51 & +0.72 \\ +1 & +0.71 & -0.51 & +0.30 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 0.076 & 0.174 & 0.174 & 0.076 & 0.076 & 0.174 & 0.174 & 0.076 \\ 0.245 & -0.245 & -0.245 & 0.245 & 0.245 & -0.245 & -0.245 & 0.245 \\ 0.245 & 0.245 & -0.245 & -0.245 & 0.245 & 0.245 & -0.245 & -0.245 \\ -0.123 & -0.296 & -0.296 & -0.123 & 0.123 & 0.296 & 0.296 & 0.123 \end{bmatrix}$$

- Efficiencies: $\begin{pmatrix} \varepsilon_I & \varepsilon_Q & \varepsilon_U & \varepsilon_V \end{pmatrix} = \begin{pmatrix} 0.93 & 0.51 & 0.51 & 0.55 \end{pmatrix}$

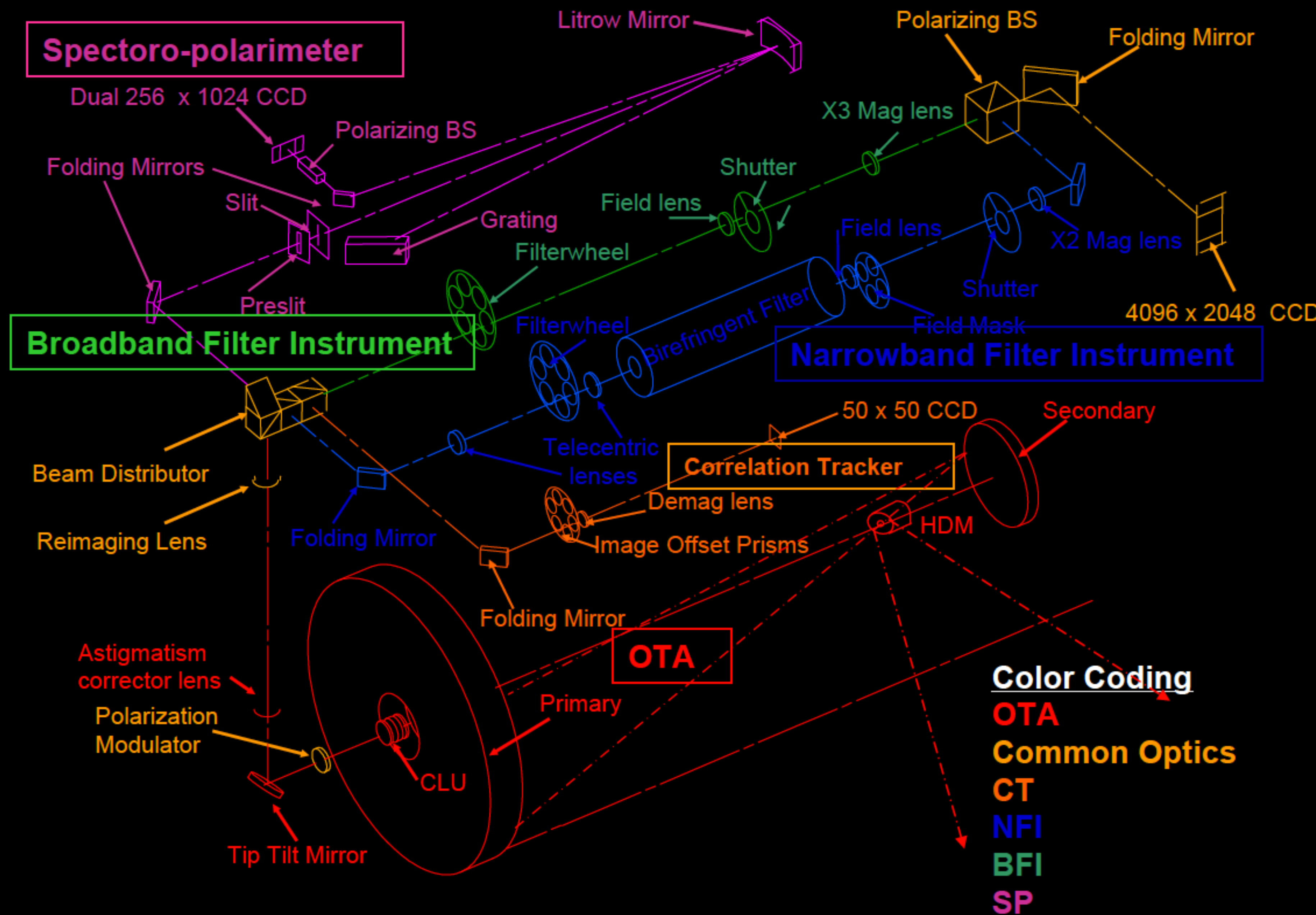
- ASP, SPINOR at DST
- SP Hinode
- 3 DKIST instruments (achromatic)
- No zeros in **D** (all contribute)
- Crystal retarders are very homogeneous
- Beam wobble is a concern



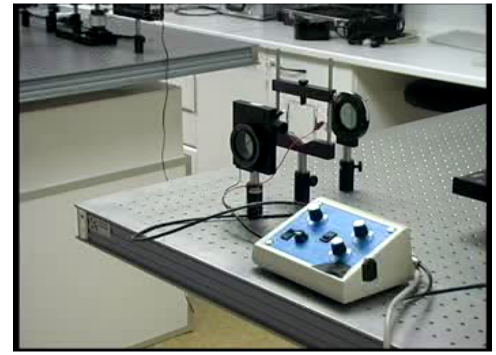
- Advanced Stokes Polarimeter

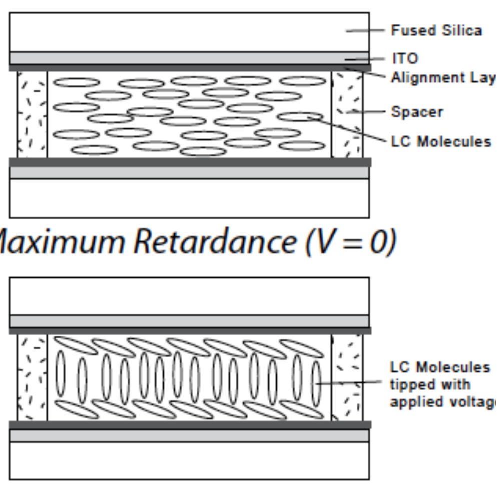


Optical layout of SOT



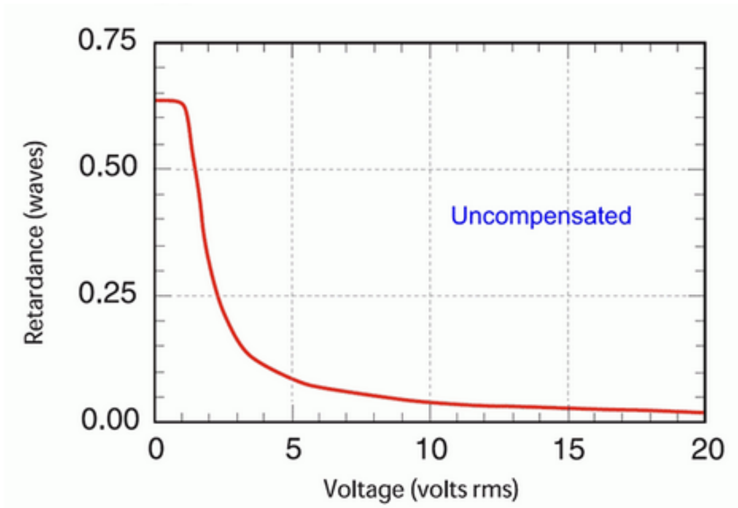
- <u>Variable Liquid Crystal Retarders</u> $\bar{\bar{M}}_{\alpha}(1,\Delta)$
- <u>Nematic</u> Liquid Crystals: change $\Delta = \Delta(V)$
- Retardance changes in a continuous way
- <u>Ferroelectric</u> Liquid Crystals: change $\alpha = \alpha(V)$
- Bistable, change in orientation typically 45°
- Temperature sensitivity in both cases
- Switching times ms (nematic) to μs (ferroelectric)
- Driving voltages DC compensated
- Degrade with UV light

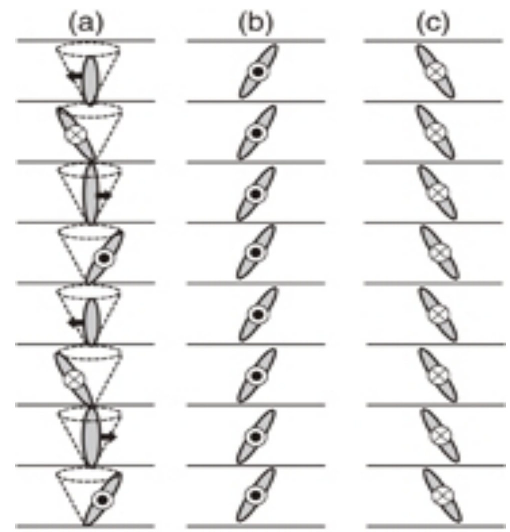




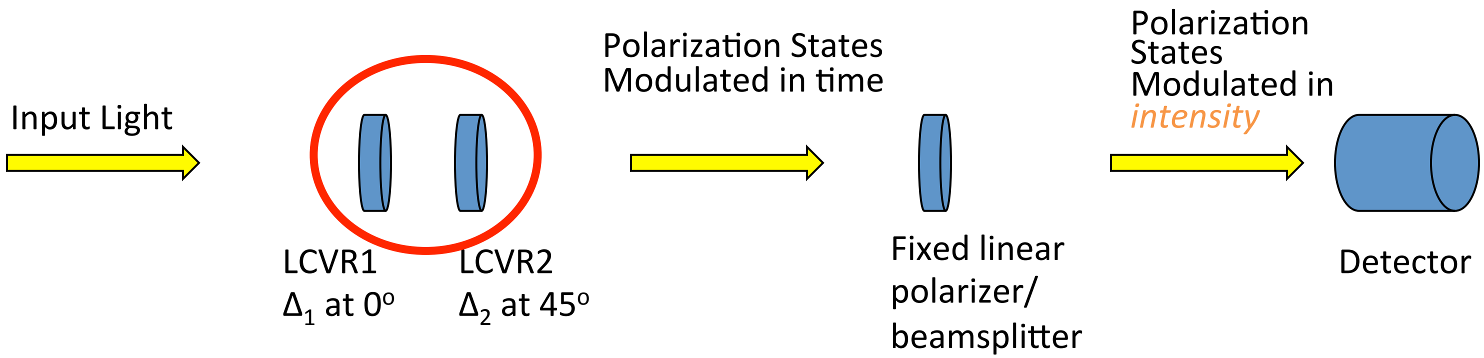
(a) Maximum Retardance (V = 0)

(b) Minimum Retardance (V >> 0)





- One LCVR at 0° followed by another at 45°:



$$\bar{\bar{M}}_0(0,0)\bar{\bar{M}}_{\frac{\pi}{4}}(1,\Delta_2)\bar{\bar{M}}_0(1,\Delta_1)=\frac{1}{2}\begin{bmatrix}1&1&0&0\\1&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&\cos\Delta_2&0&-\sin\Delta_2\\0&0&1&0\\0&\sin\Delta_2&0&\cos\Delta_2\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&\cos\Delta_1&\sin\Delta_1\\0&0&-\sin\Delta_1&\cos\Delta_1\end{bmatrix}$$

$$\bar{I}_D=\bar{\bar{M}}_0(0,0)\bar{\bar{M}}_{\frac{\pi}{4}}(1,\Delta_2)\bar{\bar{M}}_0(1,\Delta_1)=\frac{1}{2}\begin{bmatrix}1&\cos\Delta_2&\sin\Delta_2\sin\Delta_1&-\sin\Delta_2\cos\Delta_1\\1&\cos\Delta_2&\sin\Delta_2\sin\Delta_1&-\sin\Delta_2\cos\Delta_1\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{pmatrix}I_\odot\\Q_\odot\\U_\odot\\V_\odot\end{pmatrix}$$

$$I_D(t)=\frac{1}{2}(I_\odot+Q_\odot\cos\Delta_2(t)+U_\odot\sin\Delta_2(t)\sin\Delta_1(t)-V_\odot\sin\Delta_2(t)\cos\Delta_1(t))$$

- Retardance can be (almost) any → flexible
- Search for only 4 states that maximize efficiencies

$$I_D(t)=\frac{1}{2}(I_\odot+Q_\odot\cos\Delta_2(t)+U_\odot\sin\Delta_2(t)\sin\Delta_1(t)-V_\odot\sin\Delta_2(t)\cos\Delta_1(t))$$

$$|\cos\Delta_2|=|\sin\Delta_2\sin\Delta_1|=|\sin\Delta_2\cos\Delta_1|$$

- Gives the following combinations

$$\Delta_1(t_1,t_2,t_3,t_4)=\begin{bmatrix} 315, & 315, & 225, & 225 \end{bmatrix}$$

$$\Delta_2(t_1,t_2,t_3,t_4)=\begin{bmatrix} 305, & 55, & 125, & 235, \end{bmatrix}$$

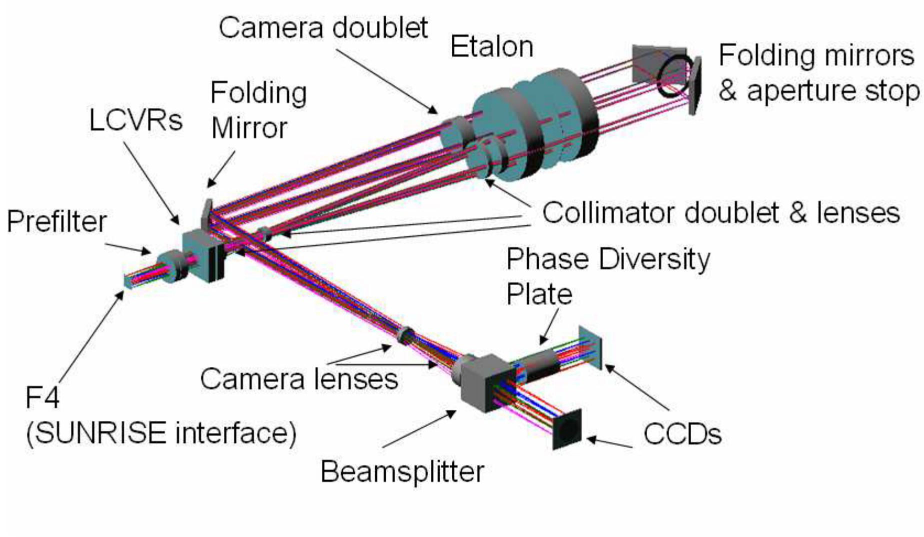
- And a modulation/demodulation matrix of

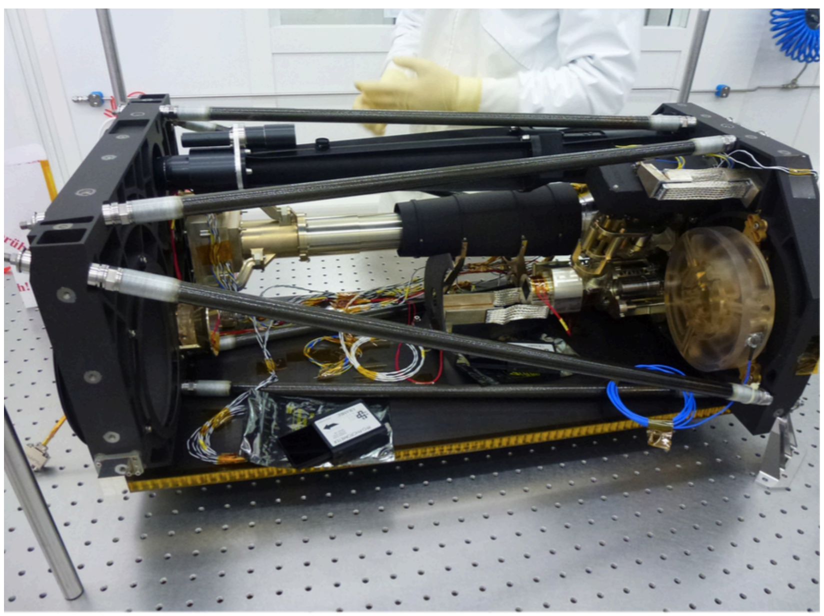
$$\mathbf{X}=\begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 1 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ 1 & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 1 & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \quad \mathbf{D}=\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} \end{bmatrix} \quad \begin{pmatrix} \varepsilon_I & \varepsilon_Q & \varepsilon_U & \varepsilon_V \end{pmatrix}=\begin{pmatrix} 1 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

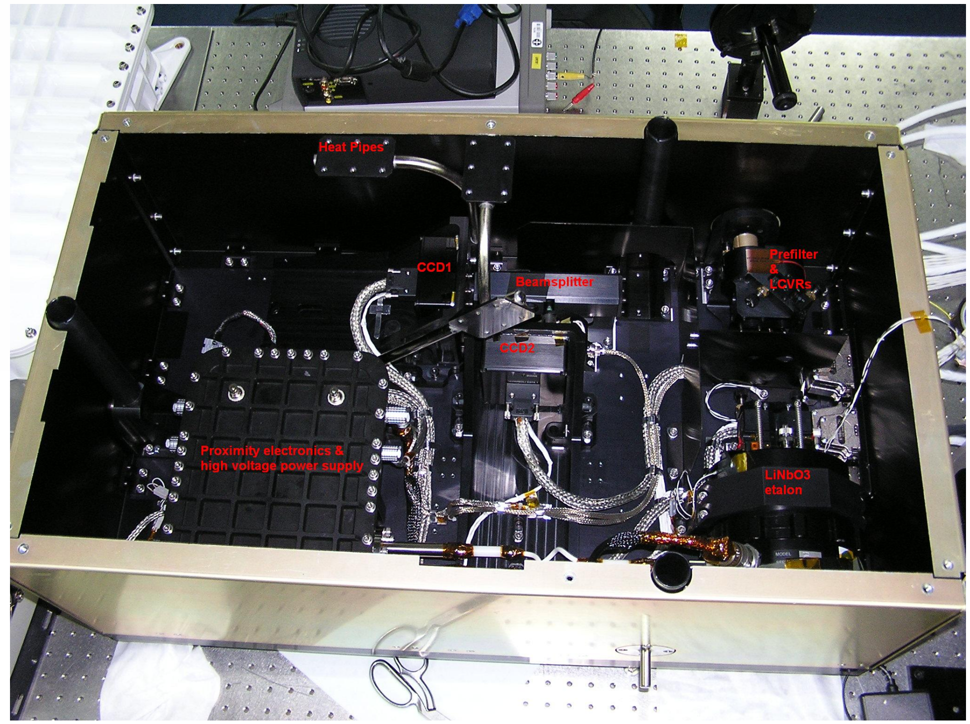
- Allow for pure states too ($I+V$ and $I-V$):

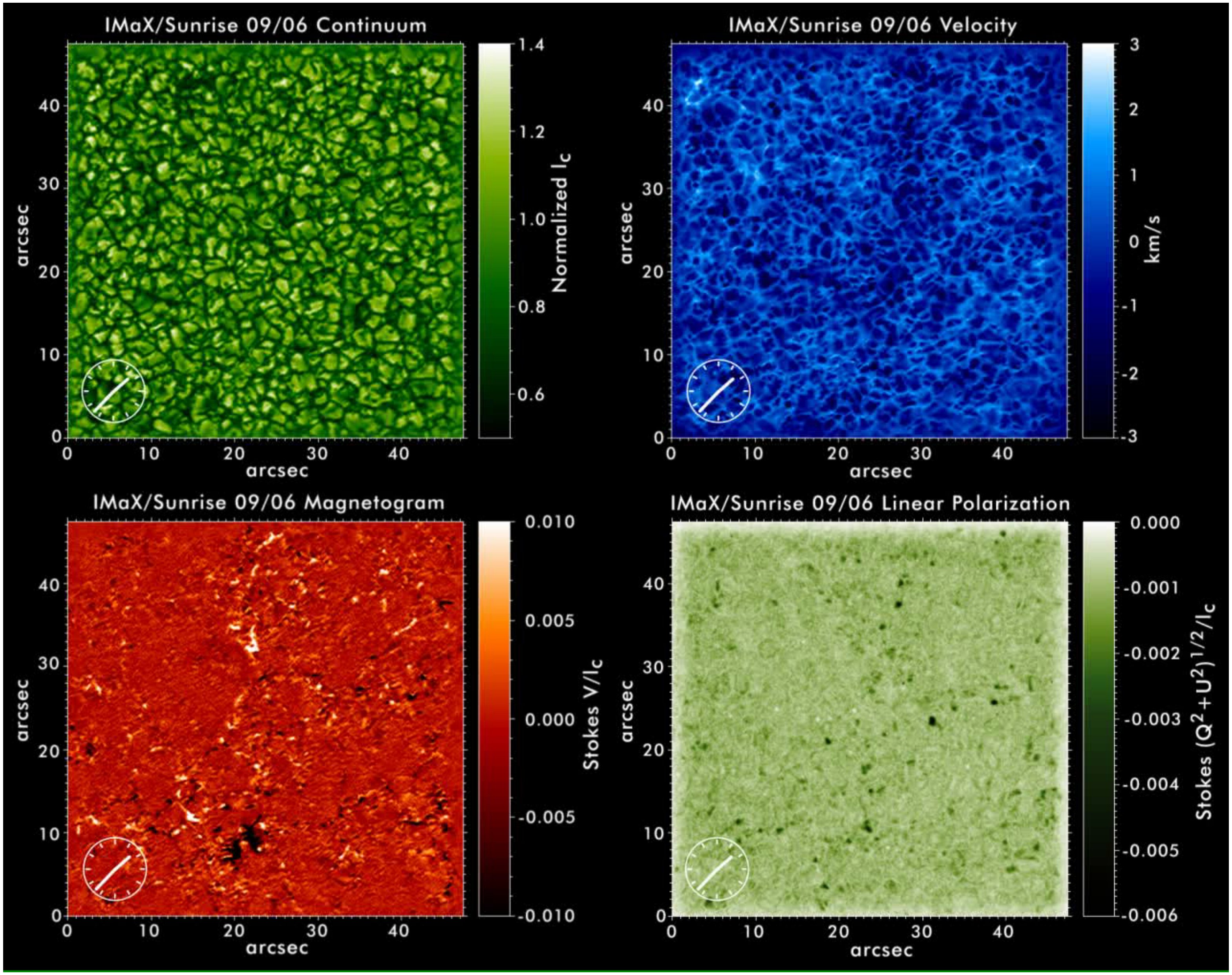
$$\Delta_1(t_1,t_2)=\left[\begin{array}{cc} 360 & 360 \end{array}\right]$$
$$\Delta_2(t_1,t_2)=\left[\begin{array}{cc} 90 & 270 \end{array}\right]$$

- 4 measures for 4 parameters
- All parameters measured all the time
- Maximum efficiencies
- LCVRs switching time are finite: degrades efficiencies
- LCVRs are not as homogeneous as crystals
- FLCs are not as flexible but are faster
- LCVRs preferred for space applications
- FLCs preferred for ground based applications
- IMaX/SUNRISE, PHI/Solar Orbiter
- VTF/DKIST


Camera doublet
Etalon
Folding mirrors
& aperture stop
Folding
Mirror
LCVRs
Prefilter
Collimator doublet & lenses
Phase Diversity
Plate
Camera lenses
F4
(SUNRISE interface)
CCDs
Beamsplitter

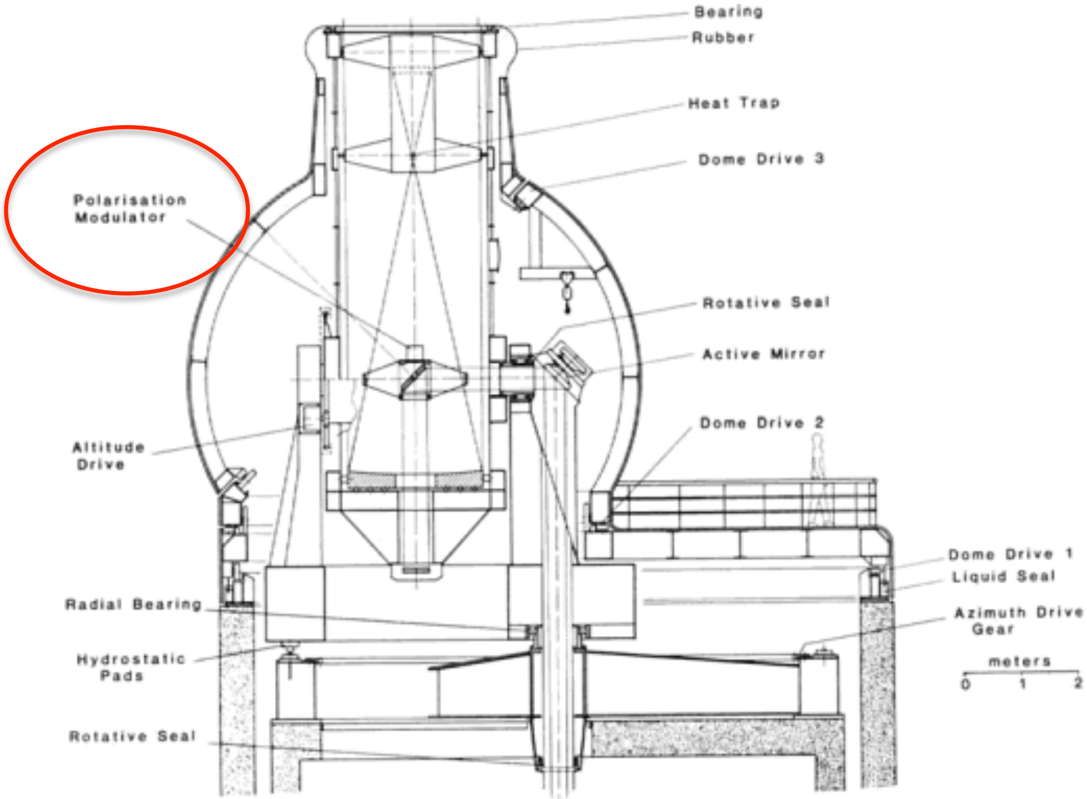


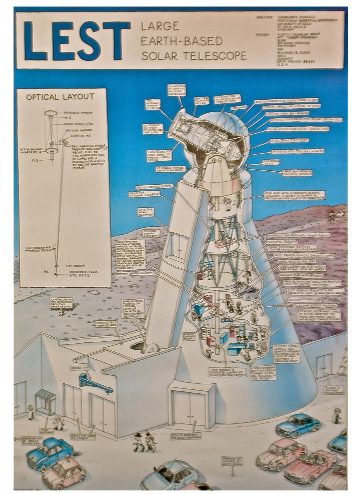

Heat Pipes
CCD1
Beamsplitter
Prefilter
&
LCVRs
CCD2
Proximity electronics &
high voltage power supply
LiNbO3
etalon



IMaX/Sunrise 09/06 Continuum
Normalized I_c
IMaX/Sunrise 09/06 Velocity
km/s
IMaX/Sunrise 09/06 Magnetogram
Stokes V/I_c
IMaX/Sunrise 09/06 Linear Polarization
Stokes $(Q^2+U^2)^{1/2}/I_c$
arcsec

Polarimetric calibration of Telescopes

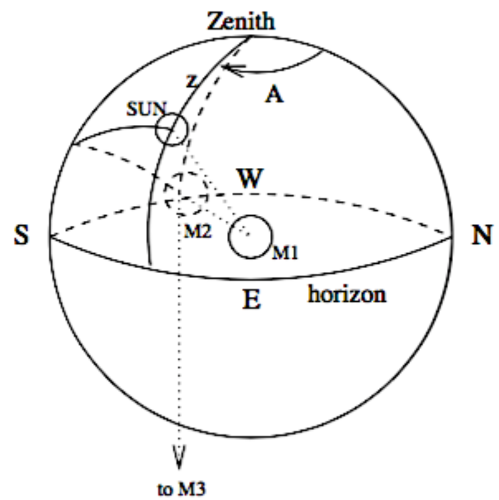


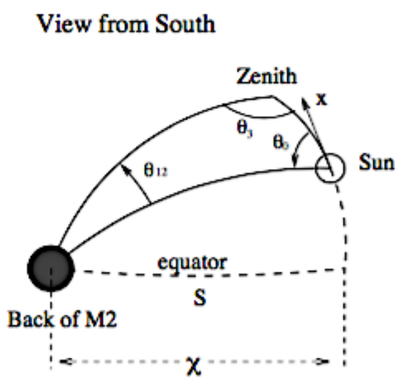


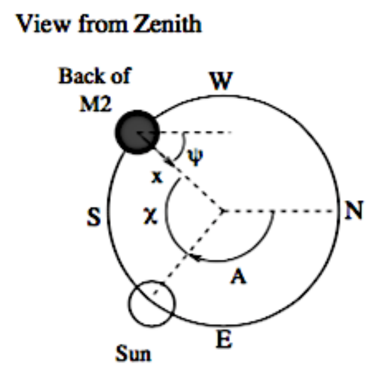
In the 90's Solar Physics was designing polarization free telescopes

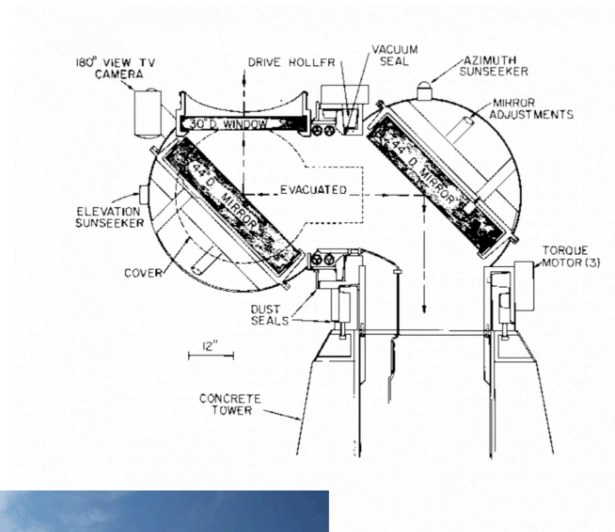
Polarimetric calibration of Telescopes

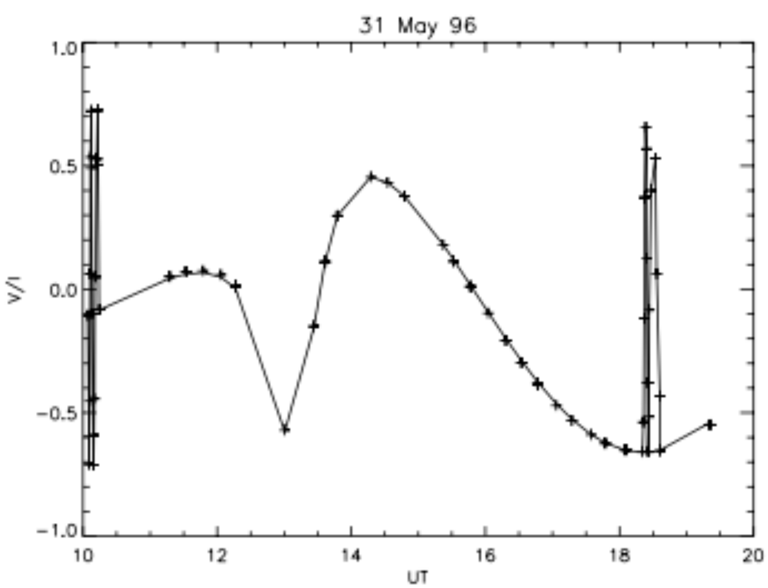
$$\bar{\bar{M}}_{SVST}=\bar{\bar{M}}_{w}(\theta_{exit},\delta_{exit})\bar{\bar{M}}_{m}(P_3,\Delta_3)\bar{\bar{R}}(\alpha)\bar{\bar{R}}(90-A)\bar{\bar{M}}_{m}(P_2,\Delta_2)R(z)\bar{\bar{M}}_{m}(P_1,\Delta_1)\bar{\bar{M}}_{w}(\theta_{ent},\delta_{ent})\bar{\bar{R}}(-90)\bar{\bar{R}}(p-P_{\odot})$$

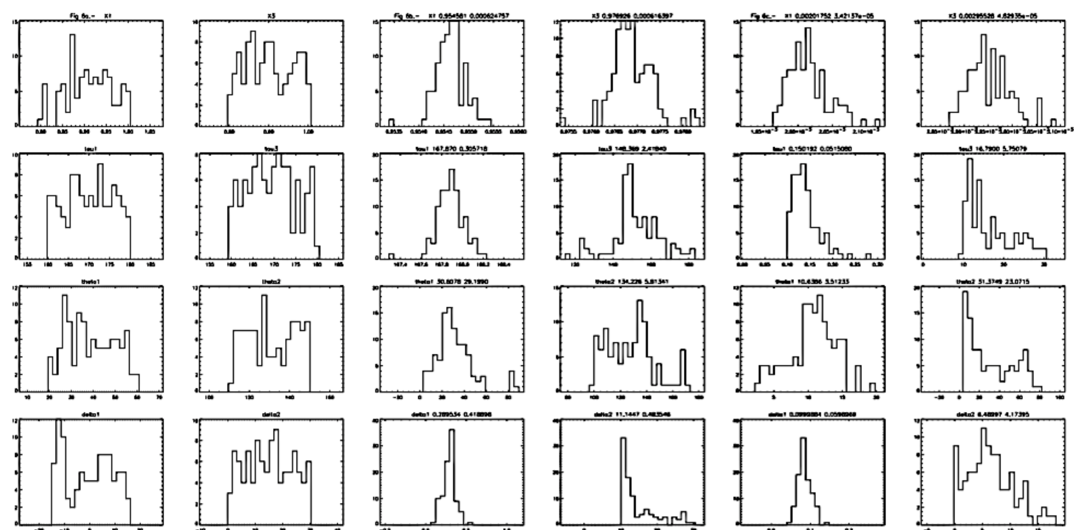






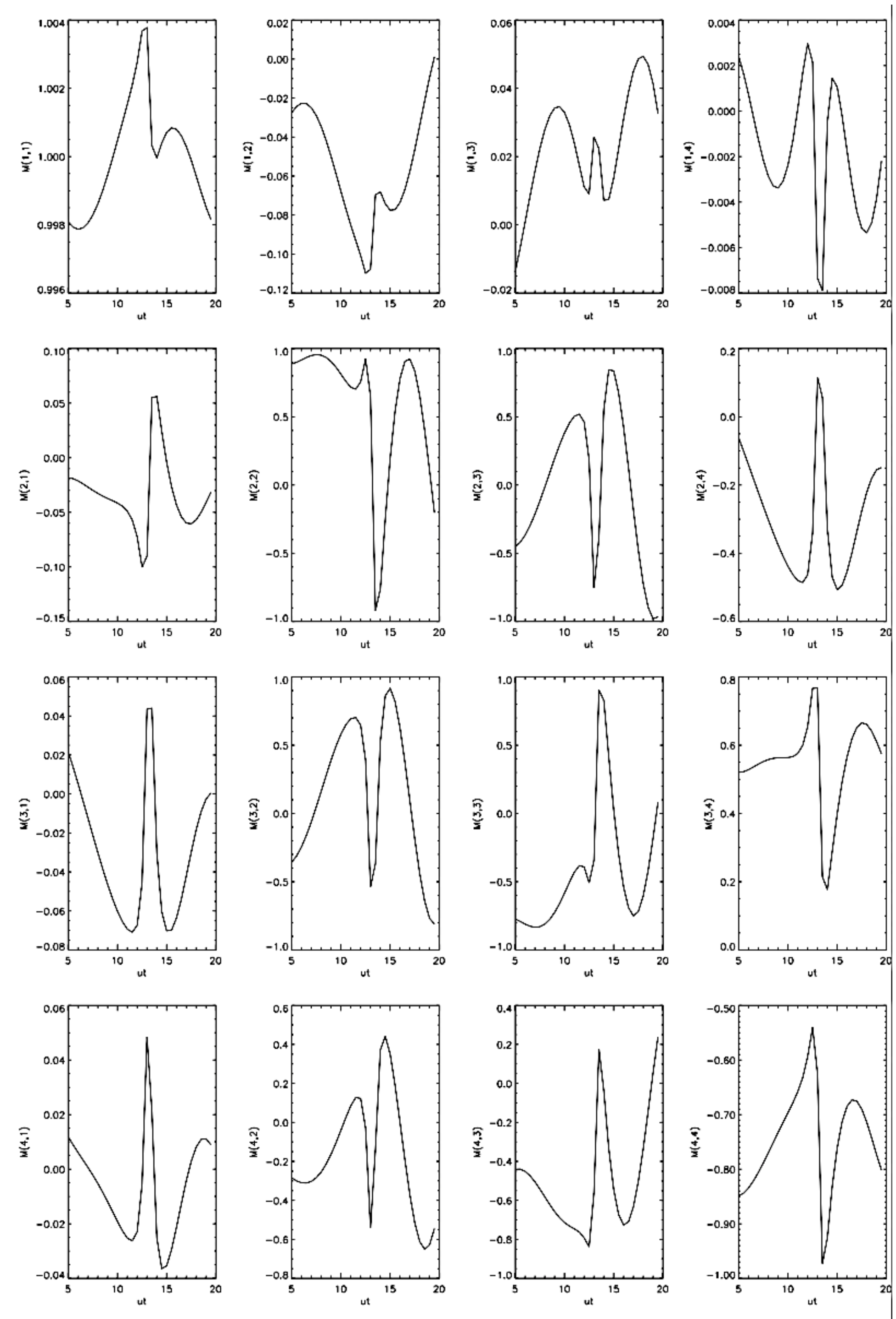




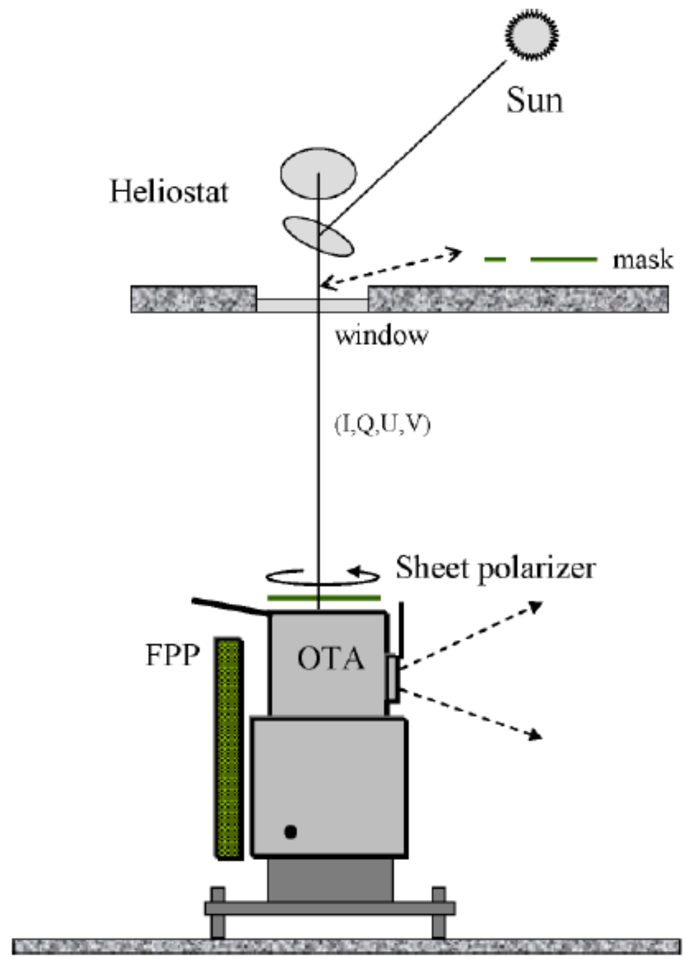


Non-linear least-square fits to a model

La Palma SVST Mueller Matrix as a function of time



Polarimetric calibration of Telescopes

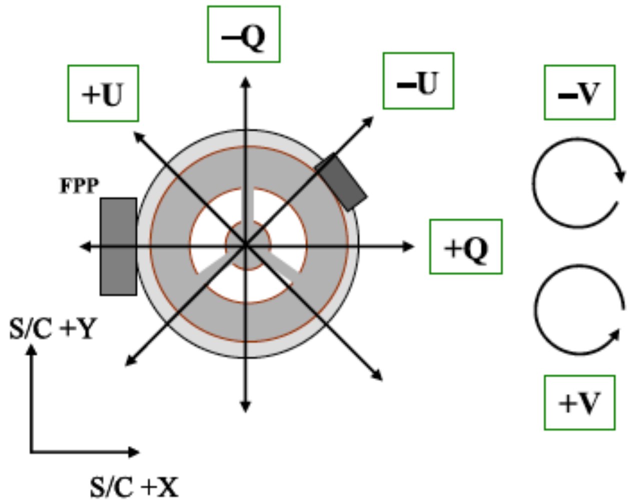


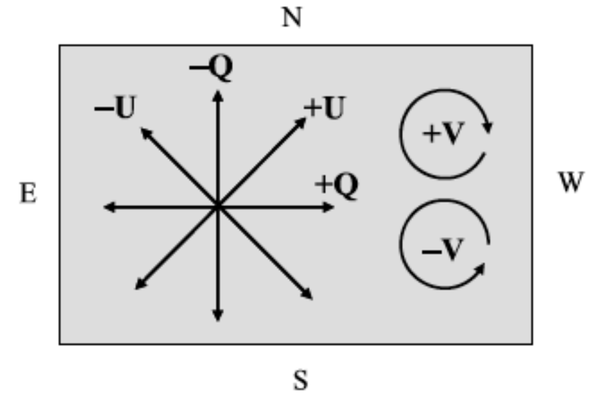
Polarimetric calibration of Telescopes

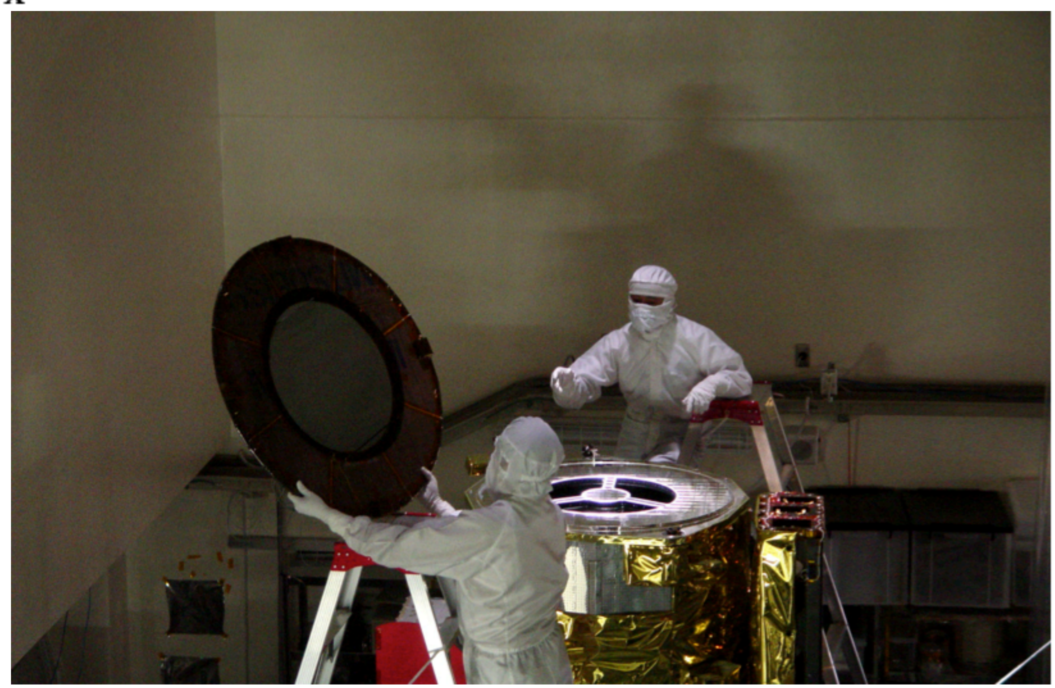
3 polarizers, 4 positions each

- Linear polarizer
- Right circular polarizer
- Left circular polarizer

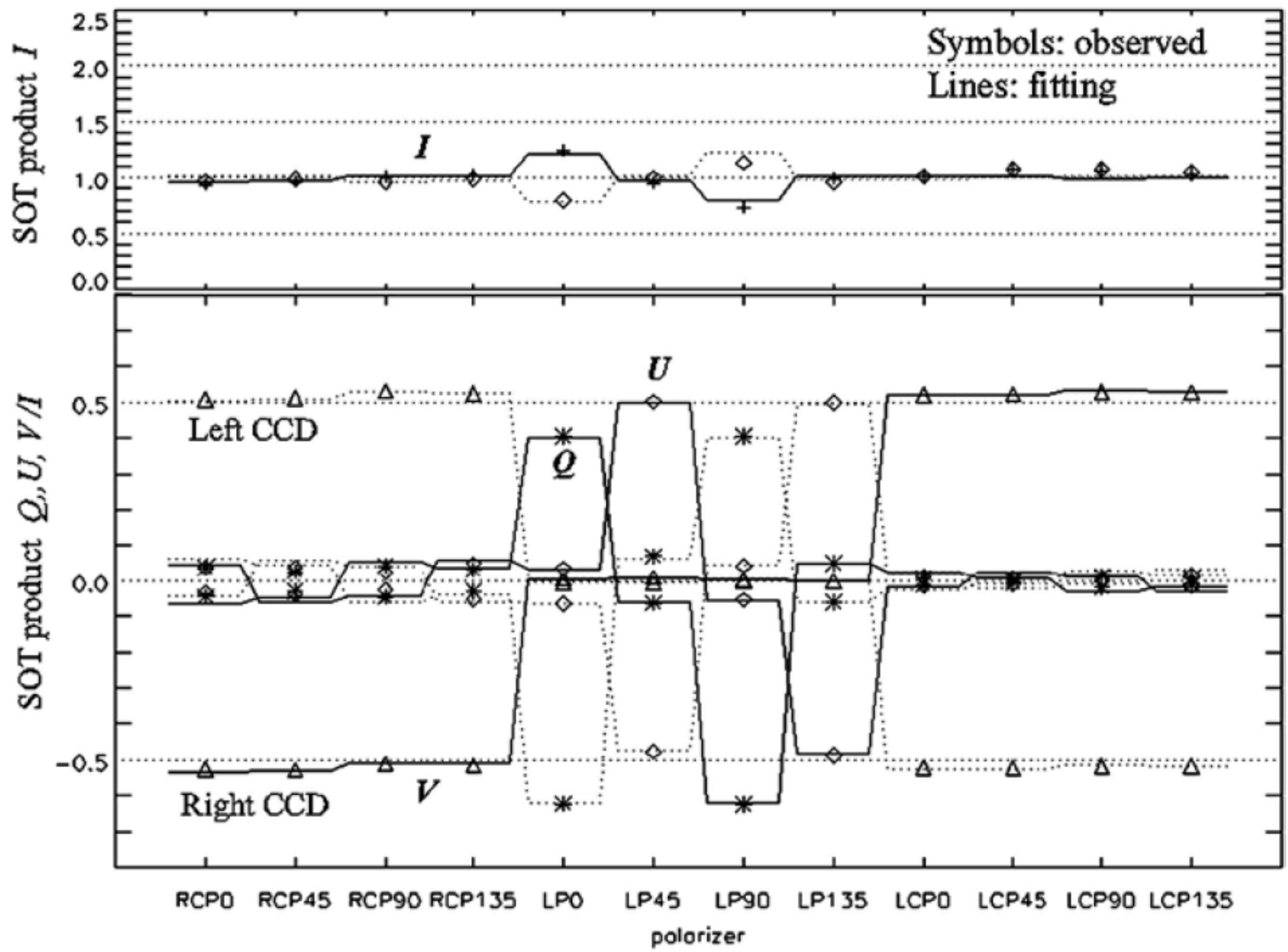
- The Mueller matrices of the sheet polarizers are known except for the two angles θ_R, θ_L of the fast axis of the circular polarizers relative to the orientation angles
- Except for these two angles, the input polarizations are therefore known



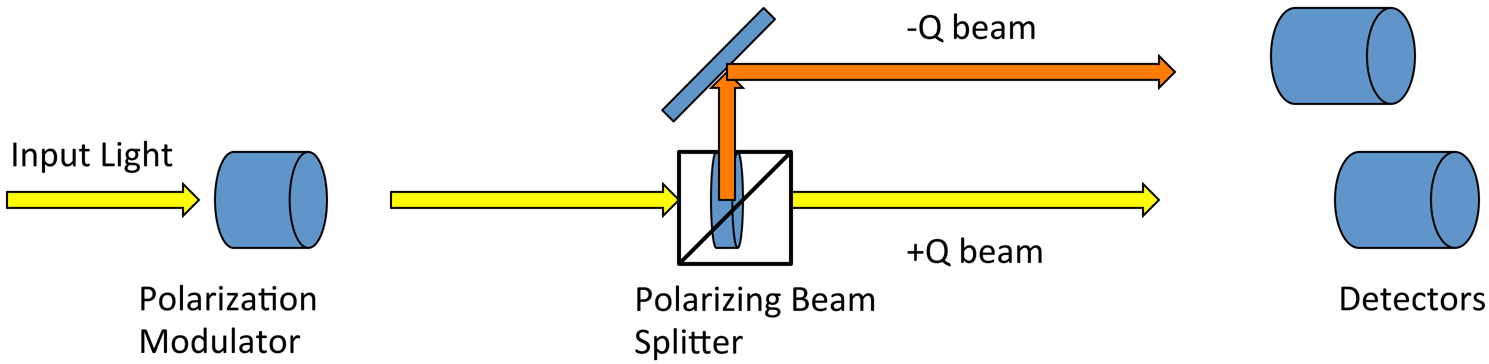




Polarimetric calibration of Telescopes



Dual Beam polarimetry



$$I_D^{+} = \frac{1}{2}\left(I_{\odot} + \left[(1+\cos\Delta) + \cos 4\omega t \cdot (1-\cos\Delta)\right]\frac{Q_{\odot}}{2} + \left[\sin 4\omega t \cdot (1-\cos\Delta)\right]\frac{U_{\odot}}{2} - \left[\sin 2\omega t \cdot \sin\Delta\right] V_{\odot}\right)$$

$$I_D^{-} = \frac{1}{2}\left(I_{\odot} - \left[(1+\cos\Delta) + \cos 4\omega t \cdot (1-\cos\Delta)\right]\frac{Q_{\odot}}{2} - \left[\sin 4\omega t \cdot (1-\cos\Delta)\right]\frac{U_{\odot}}{2} + \left[\sin 2\omega t \cdot \sin\Delta\right] V_{\odot}\right)$$

- Simplify by using $I+V$ and I-V:

$$S_1(t_1) = I + V \qquad S_2(t_1) = I - V$$

$$S_1(t_2) = I' - V' = I + \delta I - V - \delta V \qquad S_2(t_2) = I' + V' = I + \delta I + V + \delta V$$

Dual Beam polarimetry

- Each channel gives an estimate of *I* and of *V*:

$$S_1(t_1) = I + V \qquad S_2(t_1) = I - V$$

$$S_1(t_2) = I' - V' = I + \delta I - V - \delta V \qquad S_2(t_2) = I' + V' = I + \delta I + V + \delta V$$

- Merging directly $S_1(t_1)$ and $S_2(t_1)$ has flat field problems with similar outcome (not treated here):

$$I_1 = S_1(t_1) + S_1(t_2) = 2I + \delta I - \delta V \qquad I_2 = S_2(t_1) + S_2(t_2) = 2I + \delta I + \delta V$$

$$V_1 = S_1(t_1) - S_1(t_2) = 2V - \delta I + \delta V \qquad V_2 = S_2(t_1) - S_2(t_2) = -2V - \delta I - \delta V$$

- The problem are the terms δI
- If we now combine the two beam we get:

$$I = I_1 + I_2 = 4I + 2\delta I$$

$$V = V_1 - V_2 = 4V + 2\delta V$$

- Cancels δI terms
- Drastically reduces seeing/pointing spurious signals !

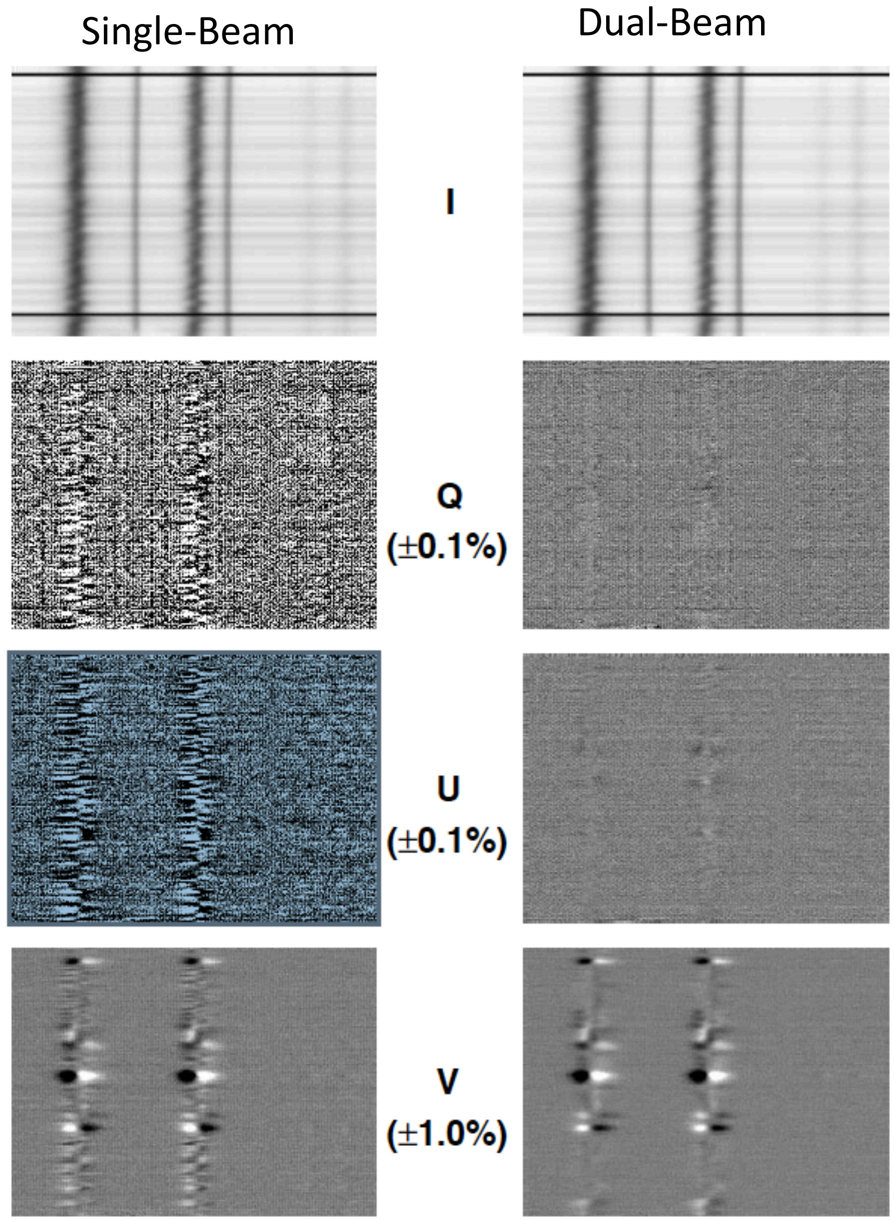


Illustration from Skumanich et al. 1997