The Radiative Transfer Equation with Polarization

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Today's Lecture

- Equation of transfer with polarization
- Solving the transfer equation in a multi-dimensional environment
- Formation height, contribution function and response function

Basic Radiative Transfer: Transport Equation

Transport along a ray:

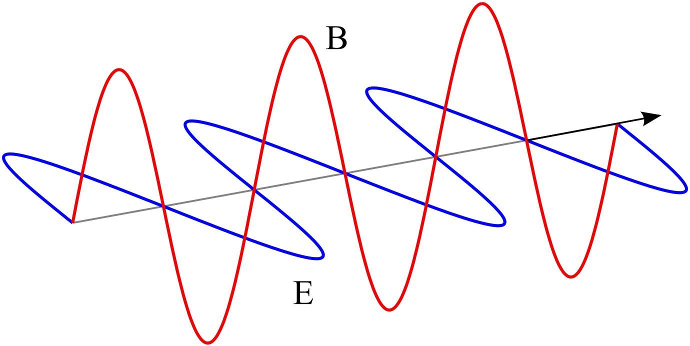
$$\frac{\mathrm{d}I_\lambda}{\mathrm{d}s} = \eta_\lambda - \chi_\lambda I_\lambda \tag{1}$$

$$\frac{\mathrm{d}I_\lambda}{\chi_\lambda \mathrm{d}s} = \frac{\mathrm{d}I_\lambda}{\mathrm{d}\tau_\lambda} = S_\lambda - I_\lambda; \qquad S_\lambda \equiv \frac{\eta_\lambda}{\chi_\lambda}$$

Integral form, the formal solution:

$$I(\tau) = I(0)e^{-\tau} + \int_0^\tau S(\tau')e^{-(\tau-\tau')}d\tau'; \quad d\tau = -\chi ds$$

A plane electromagnetic wave



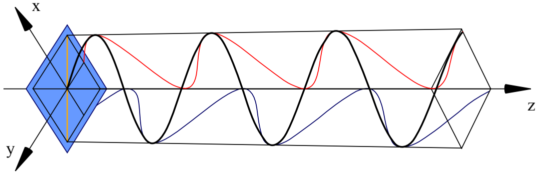
$$E(\vec{r}, t) = (A \sin(kz - \omega t), 0, 0)$$
$$B(\vec{r}, t) = (0, A \sin(kz - \omega t), 0)$$

Partially polarized light

- Most sources of electromagnetic radiation contain a large number of atoms or molecules that emit light. The orientation of the electric fields produced by these emitters may not be correlated, in which case the light is said to be unpolarized
- If there is partial correlation between the emitters, the light is partially polarized
- Partially polarized light can be described as the superposition of a completely unpolarized component, and a completely polarized one

General description of polarized light

$$E(\vec{r}, t) = (E_1 \cos(kz - \omega t), E_2 \cos(kz - \omega t + \phi), 0)$$



Linear Polarization:

$$E_1 = E_2$$
$$\phi = 0$$

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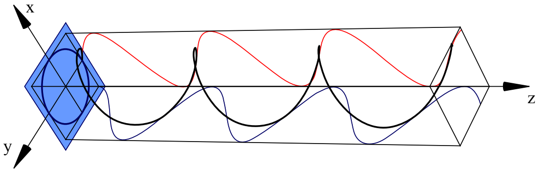
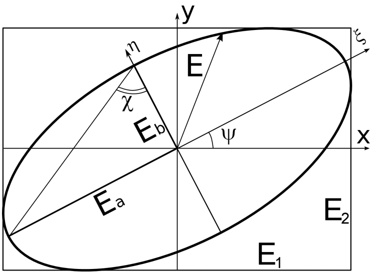


Circular Polarization:

$$E_1 = E_2$$
$$\phi = 90$$

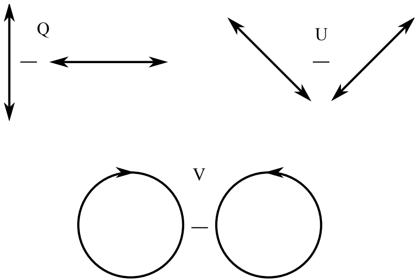
The Polarization Ellipse, General Description of Polariztion



George Gabriel Stokes, 1852

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} \equiv \begin{pmatrix} I \\ pI\cos(2\psi)\cos(2\chi) \\ pI\sin(2\psi)\cos(2\chi) \\ pI\sin(2\chi) \end{pmatrix}$$

Stokes parameters

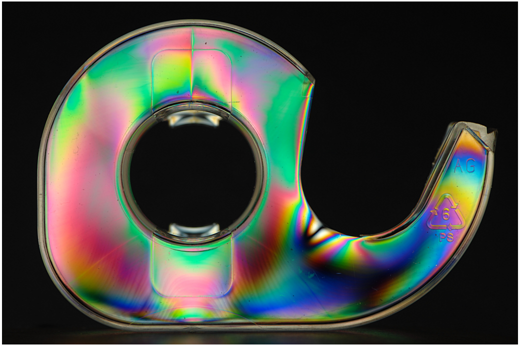


$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} E_1^2 + E_2^2 \\ E_1^2 - E_2^2 \\ 2E_1 E_2 \cos(\phi) \\ 2E_1 E_2 \sin(\phi) \end{pmatrix}$$

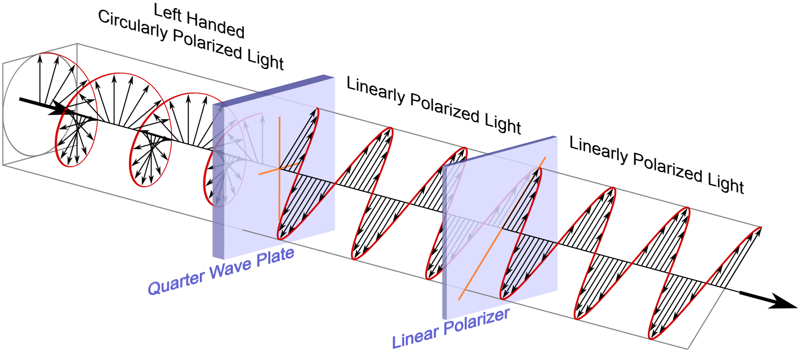
$$p \equiv \frac{\sqrt{Q^2 + U^2 + V^2}}{I^2}$$

Polarization altering materials

- Dichroic: Media in which the amplitude of waves propagating in one of the modes is reduced. Example: polarizer
- Birefringent: Media in which the two modes accrue a differential propagation delay. Example: wave plate



Linear polarizer and quarter-wave plate



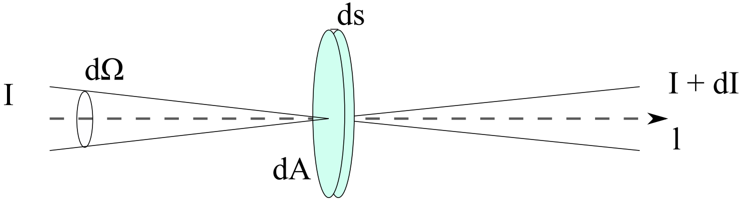
Stokes parameters give full description of polarized radiation field. Müller matrix describes interactions with materials

- The 4-element Stokes vector $S = (I, Q, U, V)$ give a full description of the intensity and polarization state of the radiation field
- Interactions of the polarized radiation field with material can be described through 4×4 matrices, the so-called Müller matrices M.

$$S_{\rm out} = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix}_{\rm out} = M \cdot S_{\rm in} = \begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix} \cdot \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix}_{\rm in}$$

- Based on constraints that follow from the process it describes the Müller matrix M has 16 elements, of which only 7 are independent

Radiative Transfer Equation

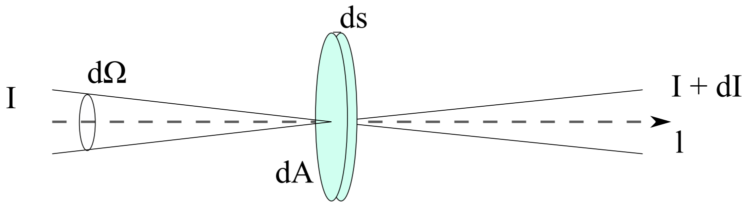


Source function for radiative transport along a ray:

$$\frac{\mathrm{d}I_\lambda}{\mathrm{d}s} = j_\lambda - k_\lambda I_\lambda$$

$$S_\lambda \equiv j_\lambda / k_\lambda$$

Equation of Polarized Radiative Transfer



Transfer Equation:

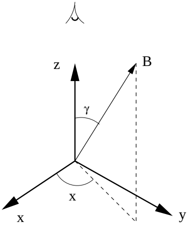
$$\frac{\mathrm{d}\mathbf{I}}{\mathrm{d}s} = -\mathbf{K}\mathbf{I} + \mathbf{j}$$

$$\mathbf{I} = (I, Q, U, V)^{\dagger}, \quad (\text{Stokes vector})$$

$$\mathbf{K} = \alpha_c \mathbf{1} + \alpha_c \mathbf{\Phi}, \quad (\text{Absorption matrix})$$

$$\mathbf{j} = (j_c + j_l \mathbf{\Phi})\mathbf{e}_0, \quad \mathbf{e}_0 = (1, 0, 0, 0)^{\dagger}$$

Line Absorption Matrix



$$\mathbf{\Phi} = \begin{pmatrix} \phi_I & \phi_Q & \phi_U & \phi_V \\ \phi_Q & \phi_I & \psi_V & -\psi_U \\ \phi_U & -\psi_V & \phi_I & \psi_Q \\ \phi_V & \psi_U & -\psi_Q & \phi_I \end{pmatrix}$$

$$\phi_I = \phi_\Delta \sin^2\gamma + \tfrac{1}{2}(\phi_+ + \phi_-), \quad \phi_\Delta = \tfrac{1}{2}\left[\phi_0 - \tfrac{1}{2}(\phi_+ + \phi_-)\right]$$

$$\phi_Q = \phi_\Delta \sin^2\gamma \cos 2\chi$$

$$\phi_U = \phi_\Delta \sin^2\gamma \sin 2\chi$$

$$\phi_V = \tfrac{1}{2}(\phi_+ - \phi_-) \cos\gamma$$

Zeeman splitting and Doppler broadening

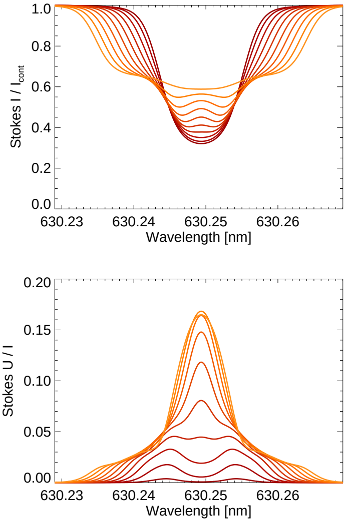
$$\phi_0 = H(a, v + v_{\rm los}); \qquad H(a, v) = \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{\exp\left(-y^2\right)}{(v - y)^2 + a^2} \mathrm{d}y$$

$$\phi_\pm = H(a, v \pm v_{\rm B} + v_{\rm los})$$

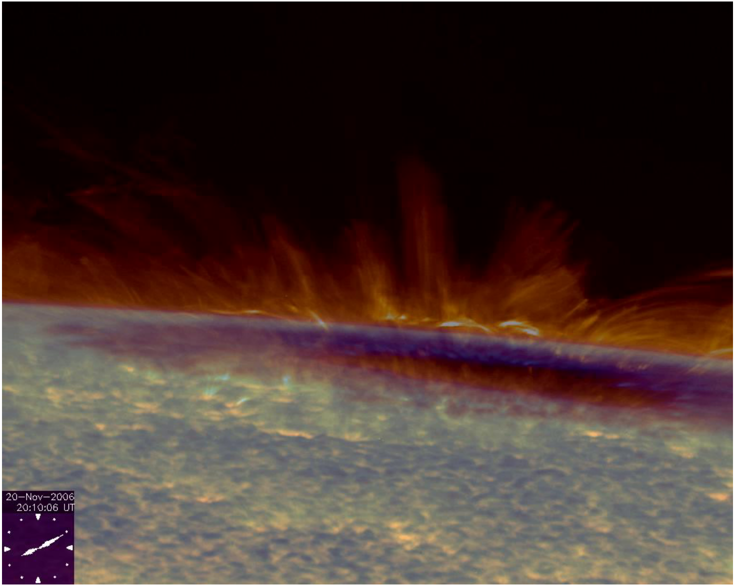
$$v_{\rm los} = \lambda \frac{\vec{v} \cdot \vec{n}}{c \Delta\lambda_{\rm D}}; \qquad \Delta\lambda_{\rm D} = \frac{v_{\rm broad}\lambda}{c}; \qquad v_{\rm broad} = \sqrt{2kT/m}$$

$$v_{\rm B} = g_{\rm L} \frac{e\lambda^2 B}{4\pi m c \Delta\lambda_{\rm D}}$$

Fe I 630.249 nm polarization for different field strengths

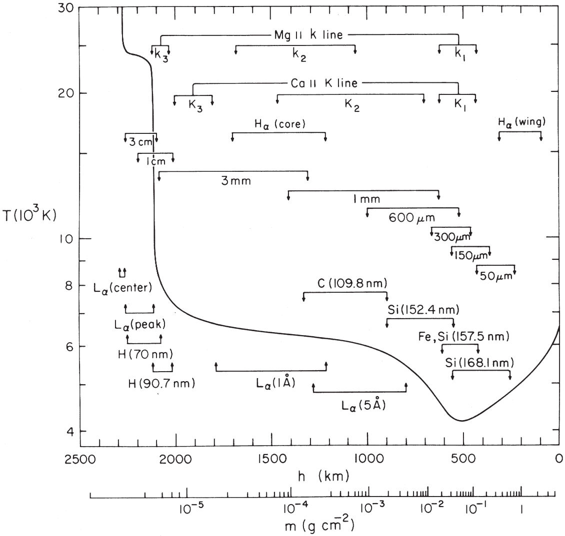


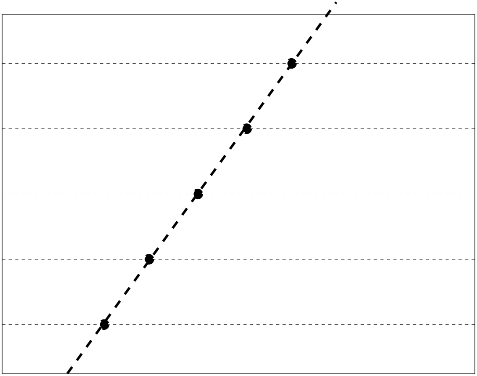
The dynamic and Inhomogeneous Solar Atmosphere



Courtesy Yukio Katsukawa, NAOJ, Japan

Hydrostatic Model FAL C (average quiet Sun)

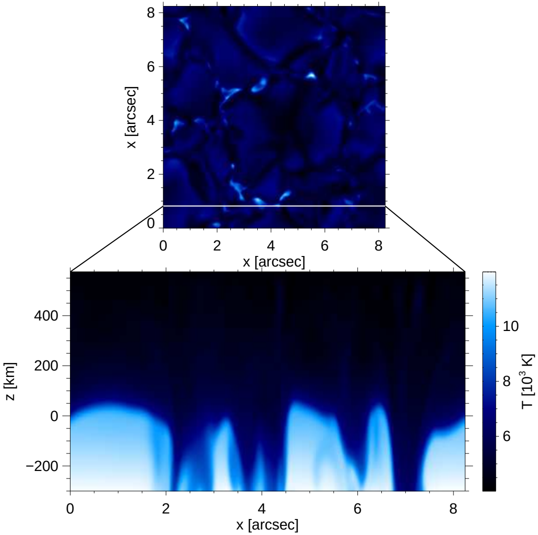




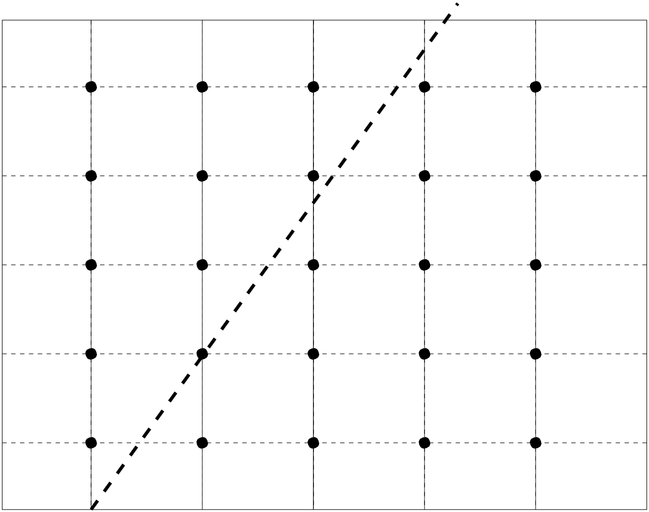
Solution to transfer equation:

$$I_\nu(\tau_\nu) = \int_0^\infty S_\nu(t) e^{-t} \mathrm{d}t$$

A vertical cross section through a 3-D Simulation



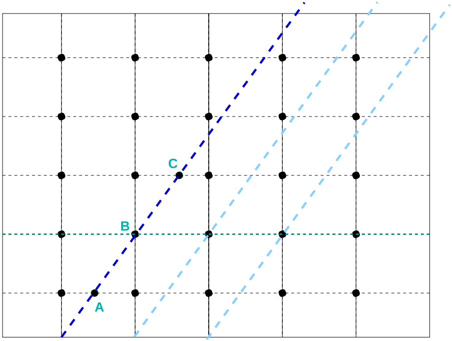
Formal solution in 2- and 3-D



If each ray is interpolated, the size of the transfer problem is of order $N^2 \times N$!

Short-Characteristics in Multi-Dimensional Geometry

Kunasz & Auer (1988),
J. Quant. Spectrosc. Radiat. Transfer, 39, 67



$$I_B = I_A e^{-\tau_{AB}} + \int_{\tau_A}^{\tau_B} S(\tau) e^{-(\tau - \tau_{AB})} \mathrm{d}\,\tau$$

Formation height of a spectral feature

How do we determine the formation height or region of influence of a spectral feauture? What is the formation height of line X?
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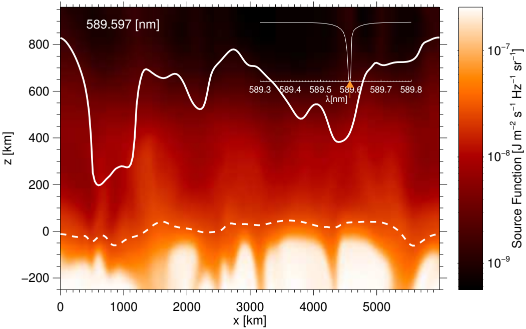
Integral form, the formal solution:

$$I(\tau) = I(0)e^{-\tau} + \int_0^\tau S(\tau')e^{-(\tau-\tau')}d\tau'; \quad d\tau = -\chi ds$$

Simplest case: Eddington – Barbier relation

Eddington – Barbier relation:

$$S = a + b\tau$$
$$I = a + b = S_\lambda(\tau = 1)$$



Contribution function

$$I(\lambda) = \int_{-\infty}^{h_0} S(\lambda, h) e^{-\tau(\lambda, h)} \left(-\frac{\mathrm{d}\tau(\lambda, h)}{\mathrm{d}h} \right) \mathrm{d}h.$$

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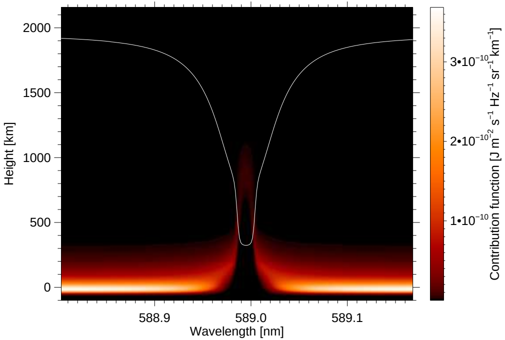
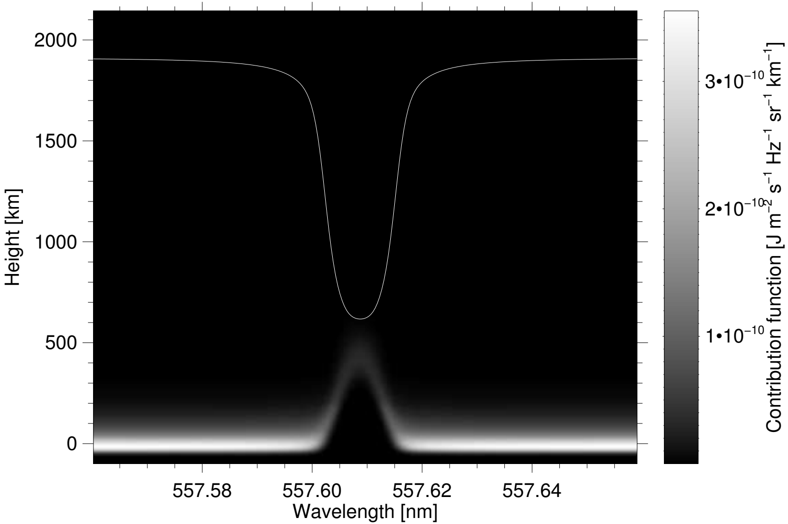


Contribution function Fe I 557.6 nm line (LTE)



Response function

Definition:

$$I(\lambda) \equiv \int_{-\infty}^{h_0} R_{I,X}(\lambda, h)\, X(h) dh$$

$$\Delta I(\lambda) = \int_{-\infty}^{h_0} R_{I,X}(\lambda, h)\, \Delta X(h) dh$$

Response function: Numerical Solution

Numerical derivation:

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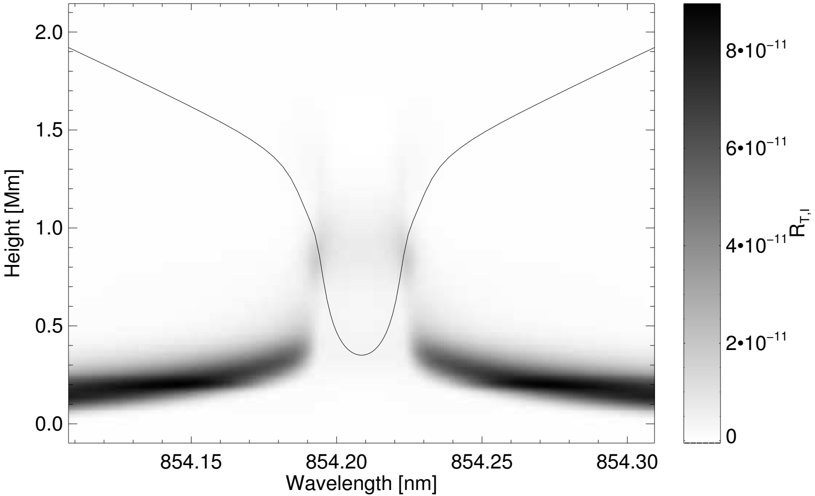
Using delta function $\Delta X(h') = x(h')\delta(h' - h)$:

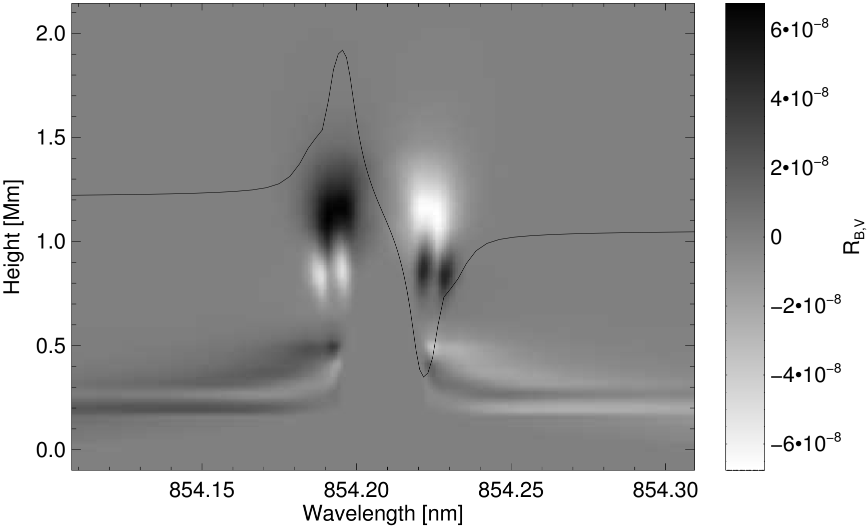
$$\Delta I^h(\lambda) = \int_{-\infty}^{h} R_{I,X}(\lambda, h')\, x(h') dh'$$

$$R_{I,X}(\lambda, h) = \frac{1}{x(h)} \Delta I^h(\lambda)$$

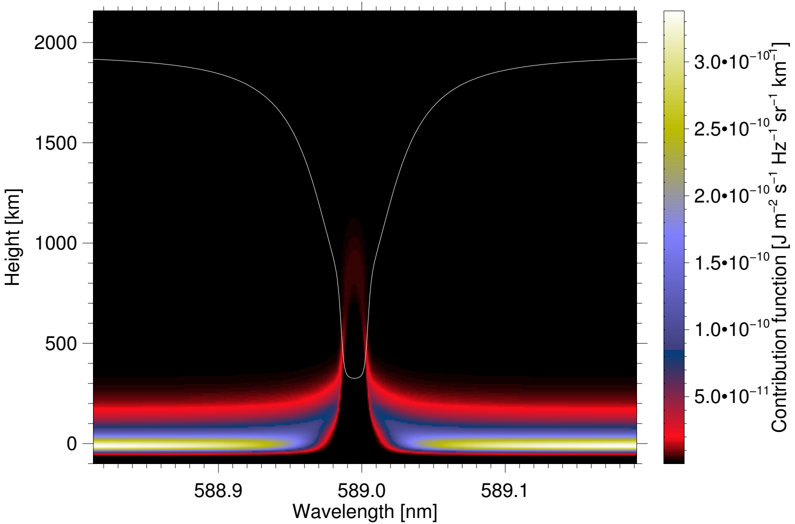
Example of Response Function: the Ca II 854.2 nm Line



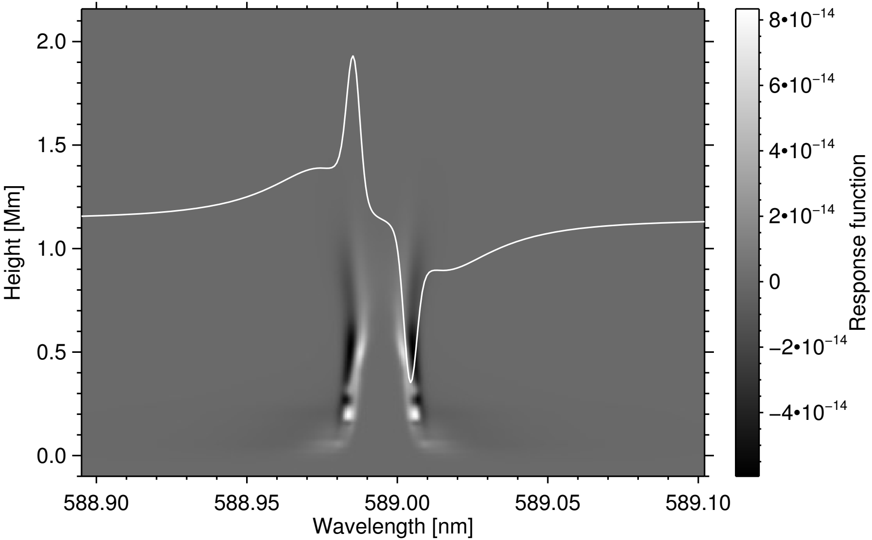
Response function Ca II 854.2 Stokes V to *B*



Contribution function Na I D_2 line



Response function Na I D_2 Stokes V to B



Conclusions

- Radiative transfer of the 4 Stokes parameters describes propagation of polarized radiation throgh a medium and its interaction with the medium.
- Polarization measurements provide additional diagnostics of astrophysical bodies through the encoding of physical properties in the polarization.
- Eddington–Barbier and contribution and response functions can be used to estimate the formation height of a spectral feature, with increasing fidelity.